Correlation of clusters: Partially truncated correlation functions and their decay.

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Abstract

In this article, we investigate partially truncated correlation functions (PTCF) of infinite continuous systems of classical point particles with pair interaction. We derive Kirkwood-Salsburg-type equations for the PTCF and write the solutions of these equations as a sum of contributions labelled by certain *forest graphs*, the connected components of which are tree graphs. We generalize the method developed by R.A. Minlos and S.K. Pogosyan (1977) in the case of truncated correlations. These solutions make it possible to derive strong cluster properties for PTCF which were obtained earlier for lattice spin systems.

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Contents

1 Introduction.

Correlation functions were first introduced in Statistical Mechanics by L.S. Ornstein and F. Zernike at the beginning of the 20th century in the study of critical fluctuations, see [25]. Mathematical studies apparently began with the work of J. Yvon [36] and the independent works of N.N. Bogolyubov [3], J.G. Kirkwood [16], and, M. Born and H.S. Green [5]. In some sense, they were completed in the works of O. Penrose [26], D. Ruelle [33], and, N.N. Bogolyubov *et al.* [4]. Correlation functions are the probability densities of correlation measures and were called m-particle distribution functions by N.N. Bogolyubov, which more accurately describes their meaning. The physical correlations between particles are in fact described by the so-called *truncated correlation functions* (TCF), or connected correlation functions, which become zero in the absence of interaction between the particles.

When studying the thermodynamic properties of statistical systems, the important characteristics are often interactions between groups of particles (the so-called clusters). Correlations between clusters are described by the so-called *partially truncated correlation functions* (PTCF), or partially connected correlation functions. In [20], J.L. Lebowitz derived bounds on the decay of correlations between two widely separated sets of particles (two point-PTCF) for ferromagnetic Ising spin systems in terms of the decay of the pair correlation. Later, in [10], some 'physically reasonable' hypotheses on the decay of the TCF and PTCF were presented and discussed. In subsequent publications of these authors [11, 12], various strong decay properties were proved for TCF of lattice and continuous systems in different situations. In [14], some general results on strong cluster properties of TCF and PTCF for lattice gases are presented (in fact, the proof of their main theorem involves long technical parts which were obtained in unpublished work of one of the authors).

In this paper, we consider classical continuous systems of point particles which interact through a two-body interaction potential. We derive equations of Kirkwood-Salsburg-type for the PTCF and apply the technique that was proposed by R.A. Minlos and S.K. Pogosyan in [22] to obtain solutions of these equations in the form of a series of contributions of certain forest diagrams. Such a representation makes it possible to obtain strong cluster properties for the PTCF in a convenient form for deriving estimates. We stress the point that explicit formulas for the upper bounds are obtained, some of which rely on some original (to our best knowledge) combinatorial identities.

2 Mathematical background.

2.1 Configuration spaces.

Let $\mathbb{R}^d$ be a d-dimensional Euclidean space, $d \geq 1$. By $\mathcal{B}(\mathbb{R}^d)$ we denote the family of all Borel sets in $\mathbb{R}^d$ and by $\mathcal{B}_c(\mathbb{R}^d)$ the system of all sets in $\mathcal{B}(\mathbb{R}^d)$ which are bounded.

The positions $\{x_i\}_{i\in\mathbb{N}}$ of identical particles are assumed to form a locally finite subset in $\mathbb{R}^d$. Because the particles are assumed to be identical, the ordering is irrelevant. Moreover, there can be more than one particle at any point. The configuration space is therefore given by locally finite maps

$$\Gamma = \Gamma_{\mathbb{R}^d} := \{\gamma : \mathbb{R}^d \to \mathbb{N}_0 \,:\, \sum_{x\in\Lambda} \gamma(x) < \infty \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\},$$

where we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, we hereafter denote by γ_Λ the restriction of γ to Λ. Further, we define the space of finite configurations Γ_0 in $\mathbb{R}^d$ as

$$\Gamma_0 := \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}, \quad \Gamma^{(n)} := \{\gamma \in \Gamma \,:\, \sum_{x\in\mathbb{R}^d} \gamma(x) = n\},$$

and the space of finite configurations in Λ as

$$\Gamma_\Lambda := \bigsqcup_{n\in\mathbb{N}_0} \Gamma_\Lambda^{(n)}, \quad \Gamma_\Lambda^{(n)} := \{\gamma \in \Gamma \,:\, \sum_{x\in\Lambda} \gamma(x) = n, \sum_{x\in\Lambda^c} \gamma(x) = 0\}.$$

The topology on Γ is generated by the subbasis $\{\mathcal{O}_K^m\}$, where $m \in \mathbb{N}_0$ and K runs over compact subsets of $\mathbb{R}^d$ with nonempty interior, given by, see, e.g., [34, Sec. 5],

$$\mathcal{O}_K^m := \{\gamma \in \Gamma \,:\, \sum_{x\in K} \gamma(x) = \sum_{x\in \mathrm{Int}(K)} \gamma(x) = m\}.$$

The topological space Γ is a polish space (i.e., metrizable, separable and complete). The corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$ is generated by the sets

$$\mathcal{W}_\Lambda^m := \{\gamma \in \Gamma \,:\, \sum_{x\in\Lambda} \gamma(x) = m\}, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d).$$

For further details, we refer the readers to [23, 21] and also the later works [17, 18].

2.2 Poisson measure on configuration spaces.

States of an *ideal gas* in equilibrium statistical mechanics are described by a *Poisson random point measure* $\pi_{z\sigma}$ on the configuration space Γ, where $z > 0$ is the activity (determining the density of particles) and σ denotes the Lebesgue measure on $\mathbb{R}^d$, i.e., $\sigma(dx) = dx$. So $\pi_{z\sigma}$ is the Poisson measure with intensity measure $z\sigma$. To define $\pi_{z\sigma}$ on Γ, we first introduce a *Lebesgue-Poisson measure* $\lambda_{z\sigma} = \lambda_{z\sigma}^\Lambda$ on the space of finite configurations Γ_Λ, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ or Γ_0, see, e.g., [23]. Given an n-tuple $(x_1, \ldots, x_n) \in \Lambda^n$, define

$$\gamma_{(x_1,\ldots,x_n)}(x) := \sum_{i=1}^n \mathbf{1}_{\{x_i\}}(x), \tag{2.1}$$

which is independent of the order of the points $x_1, \ldots, x_n$. Given a continuous function $F : \Gamma \to \mathbb{R}$, we can put $F_n(x_1, \ldots, x_n) := F(\gamma_{(x_1,\ldots,x_n)})$, $n \in \mathbb{N}$ which defines a continuous symmetric function $F_n : \mathbb{R}^{nd} \to \mathbb{R}$. Then, we define,

$$\begin{aligned}\int_{\Gamma_\Lambda} F(\gamma)\, \lambda_{z\sigma}(d\gamma) &:= \sum_{n=0}^\infty \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda F(\gamma_{(x_1,\ldots,x_n)})\, dx_1 \cdots dx_n \\ &= \sum_{n=0}^\infty \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda F_n(x_1, \ldots, x_n)\, dx_1 \cdots dx_n,\end{aligned} \tag{2.2}$$

where the term $n = 0$ in the sum is set to 1 by convention. It can be seen from (2.2) that the family of probability measures

$$\pi_{z\sigma}^\Lambda := \mathrm{e}^{-z\sigma(\Lambda)} \lambda_{z\sigma}^\Lambda, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d),$$

is consistent (i.e., forms a projective system), and by standard arguments, one can prove that there exists a unique probability measure $\pi_{z\sigma}$ on the configuration space Γ which is the projective limit of $\pi_{z\sigma}^\Lambda$.

The main feature of the measures $\pi_{z\sigma}$ and $\lambda_{z\sigma}$ is the independence of restrictions to disjoint Borel sets, which is called infinite divisibility, see, e.g., [13, Sec. 4.4]. This means that, for example, in the configuration space Γ_Λ, the following lemma holds.

Lemma 2.1 *Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $X_k \in \mathcal{B}_c(\mathbb{R}^d)$, $k = 1, 2$ such that $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = \Lambda$. Then, for all measurable functions $F_k : \Gamma_{X_k} \to \mathbb{R}$, the following identity holds*

$$\int_{\Gamma_\Lambda} F_1(\gamma) F_2(\gamma)\, \lambda_{z\sigma}(d\gamma) = \int_{\Gamma_{X_1}} F_1(\gamma)\, \lambda_{z\sigma}(d\gamma) \int_{\Gamma_{X_2}} F_2(\gamma)\, \lambda_{z\sigma}(d\gamma).$$

In [30, 28, 29] this property is the main technical tool in proving the existence of correlation functions in the infinite-volume limit. The following identity is similar, and will be used extensively.

Lemma 2.2 *Given $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and for all positive measurable functions $F : \Gamma_\Lambda \to \mathbb{R}$ and $H : \Gamma_\Lambda \times \Gamma_\Lambda \to \mathbb{R}$, the following identity holds*

$$\int_{\Gamma_\Lambda} F(\gamma) \sum_{\eta \le \gamma} H(\eta, \gamma - \eta)\, \lambda_\sigma(d\gamma) = \int_{\Gamma_\Lambda} \int_{\Gamma_\Lambda} F(\eta + \gamma) H(\eta, \gamma)\, \lambda_\sigma(d\eta)\, \lambda_\sigma(d\gamma). \tag{2.3}$$

Proof. Set $d^n x := dx_1 \cdots dx_n$. By (2.2), the left-hand side can be rewritten as

$$\begin{aligned}&\sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} F_n(x_1, \ldots, x_n) \sum_{I \subset \{1,\ldots,n\}} H_{|I|, n-|I|}(x_I, x_{I^c})\, d^n x \\ &= \sum_{n=0}^\infty \frac{1}{n!} \sum_{m=0}^n \binom{n}{m} \int_{\Lambda^n} F_n(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) H_{m,n-m}(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)\, d^n x \\ &= \sum_{m=0}^\infty \sum_{k=0}^\infty \frac{1}{m!\, k!} \int_{\Lambda^m} \int_{\Lambda^k} F_{m+k}(x_1, \ldots, x_m, y_1, \ldots, y_k) H_{m,k}(x_1, \ldots, x_m, y_1, \ldots, y_k)\, d^m x\, d^k y,\end{aligned}$$

where we set $I^c := \{1, \dots, n\} \setminus I$. It remains to use (2.2) again, and (2.3) follows. ■

2.3 Distributions in $\mathcal{D}'(\Gamma_0)$.

The space of test functions $\mathcal{D}(\Gamma_0)$ consists of functions $F : \Gamma_0 \to \mathbb{R}$ given by a sequence $(F_n)_{n\in\mathbb{N}}$ of symmetric functions $F_n \in C_0^\infty(\mathbb{R}^{dn})$ with common support such that

$$F(\gamma) = F(\gamma_{(x_1,\dots,x_n)}) = F_n(x_1, \dots, x_n), \quad \text{for any } \gamma \in \Gamma^{(n)},$$

where $\gamma_{(x_1,\dots,x_n)}$ is defined as in (2.1). Hereafter, we denote $|\gamma| := \sum_{x\in\mathbb{R}^d} \gamma(x)$, $\gamma \in \Gamma_0$.

For a given $j \in C_0^\infty(\mathbb{R}^d)$ with $|j| \le 1$, we introduce the function $\chi_j : \Gamma_0 \to \mathbb{R}$ defined as

$$\eta \mapsto \chi_j(\eta) := \begin{cases} 1, & \eta = \emptyset, \\ \prod\limits_{x\in\eta} j(x), & |\eta| \ge 1. \end{cases} \tag{2.4}$$

Here and hereafter, $x \in \eta$ means $x \in \mathbb{R}^d$ such that $\eta(x) \ge 1$. Clearly, $\chi_j \in \mathcal{D}(\Gamma_0)$.

For any $\eta \in \Gamma_0$, we define distributions δ_η such that, for any $F \in \mathcal{D}(\Gamma_0)$,

$$\langle \delta_\eta, F\rangle := \int_{\Gamma_0} \delta_\eta(\gamma) F(\gamma)\, \lambda_{z\sigma}(d\gamma) = z^{|\eta|} F(\eta). \tag{2.5}$$

In terms of 'ordinary' distributions, this means that

$$\delta_\eta(\gamma) = \begin{cases} 0, & \text{if } \ |\gamma| \ne |\eta|, \\ 1, & \text{if } \ \gamma = \eta = 0, \\ \sum\limits_{\pi\in\mathcal{S}_m} \prod\limits_{k=1}^m \delta(x_k - y_{\pi(k)}), & \text{if } \ \gamma = \gamma_{(x_1,\dots,x_m)},\, \eta = \gamma_{(y_1,\dots,y_m)}, \end{cases}$$

where $\mathcal{S}_m$ is the group of permutations of $\{1, \dots, m\}$, and the product is a direct product of δ-functions. Note that, if $\eta_1 \cdot \eta_2 = 0$ for some $\eta_1, \eta_2 \in \Gamma_0$ then δ_{η_1} and δ_{η_2} commute. Given collections $(\eta_i)_{i=1}^m$ of $\eta_i \in \Gamma_0$ with $\eta_i \cdot \eta_{i'} = 0$ if $i \ne i'$ and complex numbers $(\alpha_i)_{i=1}^m$, we can then define the product

$$\prod_{i=1}^m \Delta_{(\alpha_i,\eta_i)}(\gamma) := \prod_{i=1}^m (1 + \alpha_i \sum_{\xi_i \le \gamma} \delta_{\eta_i}(\xi_i)). \tag{2.6}$$

Note that, if $\eta_i \cdot \eta_{i'} = 0$ for $i \ne i'$, then

$$\prod_{i\in I} \sum_{\xi_i\le\gamma} \delta_{\eta_i}(\xi_i) = \sum_{\xi\le\gamma} \delta_{\sum\limits_{i\in I}\eta_i}(\xi), \quad I \subset \{1, \dots, m\}.$$

In distributional form, we have

$$\langle \prod_{i=1}^m \Delta_{(\alpha_i,\eta_i)}, F\rangle = \sum_{I\subset\{1,\dots,m\}} \prod_{i\in I} \alpha_i z^{|\eta_i|} \int_{\Gamma_0} F(\sum_{i\in I} \eta_i + \gamma)\, \lambda_{z\sigma}(d\gamma). \tag{2.7}$$

Indeed, by (2.3) (in distributional form) along with (2.5),

$$\begin{aligned} \int_{\Gamma_0} F(\gamma) \sum_{\xi\le\gamma} \delta_{\sum\limits_{i\in I}\eta_i}(\xi)\, \lambda_{z\sigma}(d\gamma) &= \int_{\Gamma_0}\int_{\Gamma_0} F(\xi+\gamma) \delta_{\sum\limits_{i\in I}\eta_i}(\xi)\, \lambda_{z\sigma}(d\xi)\, \lambda_{z\sigma}(d\gamma) \\ &= z^{\sum\limits_{i\in I}|\eta_i|} \int_{\Gamma_0} F(\sum_{i\in I}\eta_i + \gamma)\, \lambda_{z\sigma}(d\gamma). \end{aligned}$$

3 Correlation functions.

3.1 Interaction potential.

We consider a general type of two-body interaction potential

$$V_2(x, y) = \phi(|x - y|),$$

where $\phi : [0,+\infty) \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions.

(A): Assumptions about the interaction potential. *The potential ϕ is continuous on $(0,+\infty)$, $\phi(0) = +\infty$, and there exist constants $0 < r_1 < r_0 < r_2$, $\varphi_1 > 0$, $\varphi_2 > 0$, $s \geq d$ and $\varepsilon_0 > 0$ such that*

$$\phi(r) = \phi^+(r) \text{ for } 0 < r \leq r_0, \text{ and } \phi^+(r) \geq \varphi_1 r^{-s} \text{ for } r < r_1; \tag{3.1}$$

$$\phi(r) = -\phi^-(r) \text{ for } r > r_0, \text{ and } \phi^-(r) \leq \varphi_2 r^{-d-\varepsilon_0} \text{ for } r > r_2, \tag{3.2}$$

where ϕ^+ and ϕ^- denote the positive and negative parts of ϕ respectively defined as

$$\phi^+(r) := \max\{0, \phi(r)\}, \quad \phi^-(r) := -\min\{0, \phi(r)\}.$$

The shape of such potentials is illustrated in Figure 1.

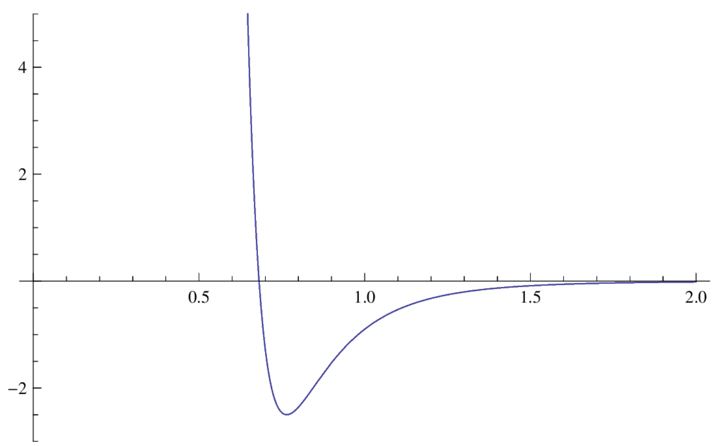

Figure 1: The Lennard-Jones potential.

A typical example is the Lennard-Jones potential, see, e.g., [33, 8], given by

$$\phi_{\mathcal{LJ}}(|x|) := \frac{\varphi_0}{|x|^6}\left(\frac{r_0^6}{|x|^6} - 1\right),$$

where $\varphi_0 > 0$ is a given constant. It is clear that the potential $\phi_{\mathcal{LJ}}$ is strongly superstable, see, e.g., [31].

Given $\eta, \gamma \in \Gamma_0$, we define the total particle interaction energy $U(\gamma)$ in the configuration γ and the interaction energy $W(\eta;\gamma)$ between the particles in the configurations η and γ respectively as

$$U(\gamma) = U_\phi(\gamma) := \sum_{\substack{\eta \leq \gamma \\ |\eta|=2}} V_2(\eta), \tag{3.3}$$

$$W(\eta;\gamma) := \sum_{\substack{x \in \eta \\ y \in \gamma}} \eta(x)\gamma(y)\phi(|x-y|). \tag{3.4}$$

Note that, under our conditions, $U(\gamma) = +\infty$ if $\gamma(x) \geq 2$ for some x, and similarly, $W(\eta;\gamma) = +\infty$ if η and γ overlap, i.e. there exist some x such that $\eta(x) \neq 0$ and $\gamma(x) \neq 0$.

Remark 3.1 *The conditions* (3.1) *and* (3.2) *are more restrictive than needed to obtain the basic expansions for the correlation functions. Sufficient assumptions to obtain analytic expansions are stability*

$$U(\gamma) \geq -B|\gamma|, \quad B \geq 0, \ \gamma \in \Gamma_0, \tag{3.5}$$

and regularity, see, e.g., [33, Sec. 4.1],

$$\nu_1(\beta) := \int_{\mathbb{R}^d} \nu_\beta(x)\, dx < +\infty, \quad \textit{with} \quad \nu_\beta(x) := |\mathrm{e}^{-\beta\phi(|x|)} - 1|. \tag{3.6}$$

We emphasize that (3.5) *and* (3.6) *hold true under the conditions* (3.1) *and* (3.2).

3.2 Gibbs measure.

With the notation introduced above, given $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the *Gibbs measure* μ_Λ on the configuration space Γ_Λ is defined as

$$\mu_\Lambda(d\gamma) := \frac{1}{Z_\Lambda} \mathrm{e}^{-\beta U(\gamma)}\, \lambda_{z\sigma}(d\gamma), \tag{3.7}$$

$$Z_\Lambda := \int_{\Gamma_\Lambda} \mathrm{e}^{-\beta U(\gamma)}\, \lambda_{z\sigma}(d\gamma), \tag{3.8}$$

where Z_Λ is the finite-volume partition function. For a survey and discussion of problems related to the construction of limit Gibbs measures for infinite systems in the space Γ, we refer the readers to the review [19] and references therein.

3.3 Correlation measure and correlation functions.

Correlation functions are the analogue of the moments of a measure. Let $\mathcal{M}^+(\mathbb{R}^d)$ denote the space of nonnegative Radon measures in $\mathcal{B}(\mathbb{R}^d)$. Consider the moments of a measure in the configuration space Γ. With every configuration $\gamma \in \Gamma$ can be associated an occupation measure according to, see, e.g., [1, 15],

$$\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} \delta_x \in \mathcal{M}^+(\mathbb{R}^d),$$

where, as previously, $x \in \gamma$ means $x \in \mathbb{R}^d$ such that $\gamma(x) \geq 1$, and, δ_x is the Dirac measure, i.e.,

$$\langle \delta_x, f\rangle = f(x), \quad f \in C_0(\mathbb{R}^d),$$

and $C_0(\mathbb{R}^d)$ denotes the space of continuous functions with compact support in $\mathbb{R}^d$.

To generalize this to the case of several variables, note that the product of distributions is not defined. For example, in the case of Gaussian measures, one usually applies Wick regularization, see, e.g., [35, 2]. An analogous procedure may be used for Poisson variables and is described below.

Let $F : \Gamma_0 \to \mathbb{R}$ be a function on the configuration space Γ_0 such that

$$F \restriction \Gamma^{(n)} := F^{(n)}(\{x_1, \dots, x_n\}) = F_n(x_1, \dots, x_n), \quad n \in \mathbb{N},$$

where, for every $n \in \mathbb{N}$, $F_n \in C_0(\mathbb{R}^{dn})$ is a symmetric function. Then,

$$\langle F^{(1)}, \gamma\rangle := \sum_{x_1\in\gamma} \langle F^{(1)}, \delta_{x_1}\rangle = \sum_{x_1\in\gamma} F_1(x_1),$$

and we define the n-th power by

$$\langle F^{(n)}, : \gamma^{\otimes n} :\rangle := \sum_{\substack{x_1,\dots,x_n\in\mathbb{R}^d \\ \gamma_{(x_1,\dots,x_n)}\leq\gamma}} F_n(x_1, \dots, x_n). \tag{3.9}$$

The *correlation measures* $\rho^{(n)}$ are defined by

$$\int_{\Gamma^{(n)}} \langle F^{(n)}, : \gamma^{\otimes n} :\rangle\, \mu(d\gamma) = \int_{\mathbb{R}^{dn}} F_n(x_1, \dots, x_n)\, \rho^{(n)}(dx_1, \dots, dx_n). \tag{3.10}$$

In case that the correlation measures $\rho^{(n)}$ are absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^{dn}$, *correlation functions* are defined as

$$\rho^{(n)}(dx_1, \dots, dx_n) := \frac{1}{n!} \rho_n(x_1, \dots, x_n)\, dx_1 \cdots dx_n.$$

These functions are obviously symmetric, so that we can write

$$\rho_n(x_1, \dots, x_n) = \rho(\eta) \restriction \Gamma^{(n)}, \quad \eta = \{x_1, \dots, x_n\}.$$

Using (3.9), (3.10) can then be rewritten in the form

$$\int_{\Gamma^{(n)}} \sum_{\substack{x_1,\ldots,x_n\in\mathbb{R}^d\\ \gamma_{(x_1,\ldots,x_n)}\leq\gamma}} F_n(x_1,\ldots,x_n)\,\mu(d\gamma) = \int_{\mathbb{R}^{dn}} F_n(x_1,\ldots,x_n)\,\rho^{(n)}(dx_1,\ldots,dx_n).$$

Given $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, we can now define the *correlation measure* ρ on the configuration space Γ_Λ by

$$\int_{\Gamma_\Lambda} F(\eta)\,\rho(d\eta) = \sum_{n=0}^{\infty} \int_{\Gamma_\Lambda^{(n)}} \sum_{\substack{x_1,\ldots,x_n\in\mathbb{R}^d\\ \gamma_{(x_1,\ldots,x_n)}\leq\gamma}} F_n(x_1,\ldots,x_n)\,\mu(d\gamma),$$

where the term $n=0$ in the sum is set to 1 by convention. In the case that the correlation measures are absolutely continuous, we have,

$$\int_{\Gamma_\Lambda} F(\eta)\rho(\eta)\,\lambda_\sigma(d\eta) = \int_{\Gamma_\Lambda} \sum_{\eta\leq\gamma} F(\eta)\,\mu(d\gamma). \tag{3.11}$$

From (3.11) along with (3.7) and (2.3), the finite-volume correlation functions can be written as

$$\rho_\Lambda(\eta) = \frac{z^{|\eta|}}{Z_\Lambda} \int_{\Gamma_\Lambda} \mathrm{e}^{-\beta U(\eta+\gamma)}\,\lambda_{z\sigma}(d\gamma), \quad \eta \in \Gamma_\Lambda. \tag{3.12}$$

Notice that $\rho_\Lambda(\eta) = 1$ whenever $\eta \in \Gamma_\Lambda^{(0)}$. Problems related to the construction of correlation functions in the infinite-volume limit are discussed in, e.g., [33, 4, 34, 30, 28, 29].

3.4 Truncated (connected) correlation functions.

Correlations between particles are better described by *truncated (connected) correlation functions* (TCF). Given $\eta \in \Gamma_0$ with $|\eta| = n \in \mathbb{N}$, these functions are defined recursively by

$$\begin{aligned} &\rho^T(x_1) := \rho(\mathbf{1}_{\{x_1\}}), \\ &\rho^T(x_1,\ldots,x_n) := \rho(\eta) - \sum_{k=2}^{n} \sum_{I_1,\ldots,I_k\subset\{1,\ldots,n\}}^{*} \prod_{l=1}^{k} \rho^T(\eta_{I_l}), \quad n\geq 2, \end{aligned} \tag{3.13}$$

where $\rho(\eta)$ are the correlation functions, and the asterisk over the sum means that the sum is over all partitions of the set $\{1,\ldots,n\}$ into k non-empty disjoint subsets, and $\eta_I = \gamma_{x_I}$. That is,

$$\sum_{l=1}^{k} \eta_{I_l} = \gamma_{(x_1,\ldots,x_n)}, \text{ with } I_l \neq \emptyset \text{ and } I_i \cap I_{i'} = \emptyset \text{ if } i \neq i'. \tag{3.14}$$

The TCF can also be written explicitly in terms of the correlation functions $\rho(\eta)$ as follows

$$\rho^T(x_1,\ldots,x_n) = \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \sum_{I_1,\ldots,I_k\subset\{1,\ldots,n\}}^{*} \prod_{l=1}^{k} \rho(\eta_{I_l}). \tag{3.15}$$

Clearly, the TCF are permutation-invariant. They can then be written as $\rho^T(\gamma_{(x_1,\ldots,x_n)})$. In case that ρ_Λ is given by (3.12), the TCF have, in the thermodynamic limit, the following representation in terms of integrals with respect to the measure $\lambda_{z\sigma}$.

Proposition 3.2 *Assume that the interaction potential ϕ satisfies* (3.5) *and* (3.6). *Then, for all $\beta > 0$ and for all $0 < z < r(\beta)$ with,*

$$r(\beta) := \mathrm{e}^{-2\beta B-1}\nu_1(\beta)^{-1}, \tag{3.16}$$

the following representation for the TCF holds true

$$\rho^T(\eta) = z^{|\eta|} \int_{\Gamma_0} \Phi^T(\eta+\gamma)\,\lambda_{z\sigma}(d\gamma), \quad \eta \in \Gamma_0. \tag{3.17}$$

Here, the function $\Phi^T(\gamma)$ is the so-called Ursell function, see, e.g., [33], given by

$$\Phi^T(\gamma) = \begin{cases} 0, & \text{if } \gamma = 0, \\ 1, & \text{if } |\gamma| = 1, \\ \sum\limits_{G\in\mathcal{G}^T(\gamma)} \prod\limits_{\{x,y\}\in\mathcal{L}(G)} C_{xy}, & \text{if } |\gamma| \geq 2, \end{cases}$$

in which $\mathcal{G}^T(\gamma)$ stands for the set of all connected graphs G (Mayer graphs) with vertices in the points x of the configuration γ, and $\mathcal{L}(G)$ is the set of all lines of the graph G, and

$$C_{xy} := \mathrm{e}^{-\beta\phi(|x-y|)} - 1. \tag{3.18}$$

Moreover, the TCF in (3.17) *can be analytically extended to the open disk of radius $r(\beta)$ given in* (3.16).

For a proof, we refer the readers to [26, 32]. See also [27] and [33, Sec. 4].
In his proof [26], O. Penrose noted that one could associate with each connected graph G on γ a unique Cayley tree obtained by deleting bonds from G in a particular way (tree graph identity). The sum over connected graphs may be rearranged by grouping together all terms (graph contributions) corresponding to a given Cayley tree, which are obtained by the procedure of "deleting". Later, D. Brydges and P. Federbush proposed in [7] a new method to derive the Mayer series for the pressure via a new type of tree graph identity. A more detailed history of the subject and some new results can be found in [24].

In this article, we derive an expansion for more general PTCF using the technique of R.A. Minlos and S.K. Pogosyan in [22] which is related to Penrose's original proof. A representation for the functions ρ^T in the form of expansions in terms of contributions from tree graphs follows as a special case.

3.5 Partially truncated (connected) correlation functions

Partially truncated (connected) correlation functions (PTCF) describe correlations between clusters of particles. Decay estimates for these correlations are an important technical tool in the proof of many physical hypotheses. For instance, see [6, Eq. (4.2)].

Given $m \in \mathbb{N}$, consider a collection $(\eta_i)_{i=1}^m$ of configurations $\eta_i \in \Gamma_0$ (for instance, resulting from the decomposition of a given $\overline{\eta} \in \Gamma_0$ into m clusters). The corresponding PTCF are defined recursively by

$$\begin{aligned} &\widetilde{\rho}^T(\eta_1) := \rho(\eta_1), \\ &\widetilde{\rho}^T(\eta_1; \ldots; \eta_m) := \rho(\sum_{i=1}^m \eta_i) - \sum_{k=2}^m \sideset{}{^*}\sum_{I_1,\ldots,I_k\subset\{1,\ldots,m\}} \prod_{l=1}^k \widetilde{\rho}^T(\widetilde{\eta}_l), \quad m \geq 2, \end{aligned} \tag{3.19}$$

where, as previously, the asterisk over the sum means that the sum is over all partitions of $\{1, \ldots, m\}$ into k non-empty disjoint subsets, and where

$$\widetilde{\eta}_l := \sum_{i\in I_l} \eta_i \quad \text{and} \quad \sum_{l=1}^k \widetilde{\eta}_l = \sum_{i=1}^m \eta_i.$$

We will sometimes use the notation $\widetilde{\rho}_m^T(\eta_1; \ldots; \eta_m) = \widetilde{\rho}^T(\eta_1; \ldots; \eta_m)$ to emphasize the number of clusters. Obviously, definition (3.19) coincides with the TCF in (3.13) when all configurations η_i consist of exactly one point. Analogous to (3.15), the PTCF can be expressed directly in terms of the $\rho(\tilde{\eta}_i)$ as

$$\widetilde{\rho}^T(\eta_1; \ldots, \eta_m) = \sum_{k=1}^m (-1)^{k-1}(k-1)! \sideset{}{^*}\sum_{I_1,\ldots,I_k\subset\{1,\ldots,m\}} \prod_{l=1}^k \rho(\tilde{\eta}_l). \tag{3.20}$$

To derive such an expression for the PTCF, we introduce a generating functional. It is a generalization of the generating functional introduced in [14] for spin systems.

For a given nonnegative $j \in C_0^\infty(\mathbb{R}^d)$, define the smoothed correlation function ρ_j by

$$\rho_j(\eta) = \rho_{j;1}(\eta) := \frac{z^{|\eta|}}{Z_j} \int_{\Gamma_0} \chi_j(\eta + \gamma) \mathrm{e}^{-\beta U(\eta+\gamma)}\, \lambda_{z\sigma}(d\gamma), \quad \eta \in \Gamma_0, \tag{3.21}$$

where the function $\chi_j : \Gamma_0 \to \mathbb{R}_+$ is defined as in (2.4), and

$$Z_j := \int_{\Gamma_0} \chi_j(\gamma) \mathrm{e}^{-\beta U(\gamma)}\, \lambda_{z\sigma}(d\gamma).$$

Using the definitions (2.5) and (2.6) with nonnegative reals $(\alpha_i)_{i=1}^m$, we now put

$$\widetilde{F}^T_{\rho_j}(\alpha, \eta)_1^m := \log(Z_j((\alpha_i, \eta_i)_{i=1}^m)), \tag{3.22}$$

where

$$Z_j((\alpha_i, \eta_i)_{i=1}^m) := \langle \prod_{i=1}^m \Delta_{(\alpha_i,\eta_i)}, \chi_j \mathrm{e}^{-\beta U} \rangle = \int_{\Gamma_0} \prod_{i=1}^m \Delta_{(\alpha_i \eta_i)}(\gamma) \chi_j(\gamma) \mathrm{e}^{-\beta U(\gamma)}\, \lambda_{z\sigma}(d\gamma). \tag{3.23}$$

Note that, if $j(x) = \mathbf{1}_\Lambda(x)$, where $\mathbf{1}_\Lambda$ denotes the indicator function of a set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, and if $\alpha_i = 0$, $i = 1, \ldots, m$, then (3.23) reduces to the partition function in (3.8). Further, define

$$\widetilde{\rho}^T_{j;r}(\eta_1; \ldots; \eta_r | (\alpha_i, \eta_i)_{i=r+1}^m) := (\prod_{i=1}^r \frac{\partial}{\partial \alpha_i}) \widetilde{F}^T_{\rho_j}((\alpha_i, \eta_i)_{i=1}^m)\big|_{\alpha_1 = \cdots = \alpha_r = 0}, \quad 1 \le r \le m. \tag{3.24}$$

We call *r-point j-PTCF*, or simply *j-PTCF* when $r = m$, the following functions

$$\widetilde{\rho}^T_{j;r}(\eta_1; \ldots; \eta_r) := \widetilde{\rho}^T_{j;r}(\eta_1; \ldots; \eta_r | (\alpha_i, \eta_i)_{i=r+1}^m)\big|_{\alpha_{r+1} = \cdots = \alpha_m = 0}. \tag{3.25}$$

We conclude this section with the following lemma

Lemma 3.3 *Given $r, m \in \mathbb{N}$ such that $1 \le r \le m$, the r-point j-PTCF associated to the collection $(\eta_i)_{i=1}^m$ of configurations $\eta_i \in \Gamma_0$ are given by*

$$\widetilde{\rho}^T_{j;r}(\eta_1; \ldots; \eta_r) = \sum_{k=1}^r (-1)^{k-1}(k-1)! \sum^*_{\{J_1, \ldots, J_k\} \subset \{1, \ldots, r\}} \prod_{l=1}^k \rho_j(\sum_{i \in J_l} \eta_i), \tag{3.26}$$

where the second sum in (3.26) runs over all partitions of $\{1, \ldots, r\}$ into k non-empty subsets $J_1, \ldots, J_k$ with the restrictions (3.14). In particular, when $j(x) = \mathbf{1}_\Lambda(x)$ the functions (3.25) correspond to the finite-volume PTCF in Λ, and when $j(x) = 1$ they correspond to the PTCF in $\mathbb{R}^d$.

Remark 3.4 *One can show by induction that, for $r \ge 2$, $\widetilde{\rho}^T_{j;r}(\eta_1; \ldots; \eta_r) = 0$ if there exists $i_0 \in \{1, \ldots, r\}$ such that $|\eta_{i_0}| = 0$, see (3.26) along with (3.21).*

Proof. The key ingredient is the following formula. Given a smooth function $Z : \mathbb{R}^m \to (0, +\infty)$,

$$(\prod_{i=1}^k \frac{\partial}{\partial \alpha_i}) \log(Z((\alpha_i)_{i=1}^m)) =$$
$$\sum_{k=1}^r (-1)^{k-1}(k-1)! \sum^*_{\{J_1, \ldots, J_k\} \subset \{1, \ldots, r\}} \prod_{l=1}^k \frac{1}{Z((\alpha_i)_{i=1}^m)} (\prod_{i \in J_l} \frac{\partial}{\partial \alpha_i}) Z((\alpha_i)_{i=1}^m), \quad 1 \le r \le m, \tag{3.27}$$

which easily follows by induction. On the other hand, from (3.23) along with (2.7), we have,

$$(\prod_{i \in J_l} \frac{\partial}{\partial \alpha_i}) Z_j((\alpha_i, \eta_i)_{i=1}^m) =$$
$$z^{\sum_{i \in J_l} |\eta_i|} \sum_{I \subset \{1, \ldots, m\} \setminus J_l} \prod_{i \in I} (\alpha_i z^{|\eta_i|}) \int_{\Gamma_0} \chi_j(\sum_{i \in J_l} \eta_i + \sum_{i \in I} \eta_i + \gamma) \mathrm{e}^{-\beta U(\sum_{i \in J_l} \eta_i + \sum_{i \in I} \eta_i + \gamma)}\, \lambda_{z\sigma}(d\gamma).$$

Setting the remaining $\alpha_i = 0$, only the empty set $I = \emptyset$ survives. In view of (3.21), we then obtain

$$\frac{1}{Z_j((\alpha_i, \eta_i)_{i=1}^m)} (\prod_{i \in J_l} \frac{\partial}{\partial \alpha_i}) Z_j((\alpha_i, \eta_i)_{i=1}^m)\big|_{\alpha_1 = \cdots \alpha_m = 0} = \rho_j(\sum_{i \in J_l} \eta_i). \tag{3.28}$$

Replacing $Z((\alpha_i)_{i=1}^m)$ by $Z_j((\alpha_i, \eta_i)_{i=1}^m)$ in (3.27), (3.26) follows from (3.28). ■

In particular, taking the limit $j \to 1$ in (3.26) with $k = m$, we obtain (3.20).

4 Equations for PTCF and their solutions.

4.1 Kirkwood-Salsburg-type equations.

Let $(\eta_i)_{i=1}^m$, $m \geq 2$ be a collection of configurations in Γ_0 such that $\sum_{i=1}^m |\eta_i| > 0$. We start by deriving Kirkwood-Salsburg-type equations for the 1-point j-PTCF. Assume that $|\eta_1| > 0$ (we may change the cluster labelling if needed). From (3.25)–(3.24) (with $k = 1$) and (3.22)–(3.23), the 1-point j-PTCF reads

$$\tilde{\rho}^T_{j;1}(\eta_1|(\alpha_i,\eta_i)_{i=2}^m) = \frac{1}{Z_j((\alpha_i,\eta_i)_{i=1}^m)} \frac{\partial}{\partial \alpha_1} \int_{\Gamma_0} \prod_{i=1}^m \Delta_{(\alpha_i \eta_i)}(\gamma)\chi_j(\gamma) \mathrm{e}^{-\beta U(\gamma)}\, \lambda_{z\sigma}(d\gamma)\big|_{\alpha_1=0}. \tag{4.1}$$

In view of (3.4), let $x_1 \in \eta_1$ such that

$$\widehat{W}(\eta_1) := W(\mathbf{1}_{\{x_1\}}; \eta_1 - \mathbf{1}_{\{x_1\}}) \geq -2B,$$

where $B \geq 0$ is defined by (3.5). Note that the existence of such a point in any configuration follows from (3.5), see [33, Chap. 4]. Note also that $\widehat{W}(\eta_1) = +\infty$ if $\eta_1(x_1) > 1$. Consider the decomposition,

$$\mathrm{e}^{-\beta U(\eta_1+\gamma)} = \mathrm{e}^{-\beta \widehat{W}(\eta_1)} \mathrm{e}^{-\beta W(x_1;\gamma)} \mathrm{e}^{-\beta U(\eta_1'+\gamma)} = \mathrm{e}^{-\beta \widehat{W}(\eta_1)} \sum_{\xi \leq \gamma} K(x_1;\xi) \mathrm{e}^{-\beta U(\eta_1'+\gamma)}, \quad \gamma \in \Gamma_0, \tag{4.2}$$

where $\eta_1' := \eta_1 - \mathbf{1}_{\{x_1\}}$, and

$$K(x_1;\xi) := \prod_{y\in\xi} C_{x_1 y} = \prod_{y\in\xi} (\mathrm{e}^{-\beta\phi(|x_1-y|)} - 1).$$

Inserting the right-hand side of the second equality of (4.2) into (4.1), we have,

$$\tilde{\rho}^T_{j;1}(\eta_1|(\alpha_i,\eta_i)_{i=2}^m) = \frac{z^{|\eta_1|}\mathrm{e}^{-\beta\widehat{W}(\eta_1)}}{Z_j((\alpha_i,\eta_i)_{i=2}^m)} \int_{\Gamma_0} \sum_{\xi\leq\gamma} K(x_1;\xi) \prod_{i=2}^m \Delta_{(\alpha_i\eta_i)}(\gamma)\chi_j(\eta_1+\gamma)\mathrm{e}^{-\beta U(\eta_1'+\gamma)}\, \lambda_{z\sigma}(d\gamma). \tag{4.3}$$

Putting $\alpha_2 = \cdots = \alpha_m = 0$ in (4.3) and then using (2.3) (extended to the configuration space Γ_0) along with the identity $\tilde{\rho}^T_{j;1}(\eta_1' + \gamma) = \rho_j(\eta_1' + \gamma)$, we obtain the Kirkwood-Salsburg equation

$$\tilde{\rho}^T_{j;1}(\eta_1) = \rho_j(\eta_1) = z\mathrm{e}^{-\beta\widehat{W}(\eta_1)} j(x_1) \int_{\Gamma_0} K(x_1;\xi)\rho_j(\eta_1'+\xi)\, \lambda_\sigma(d\xi). \tag{4.4}$$

We now generalize those equations for the m-point j-PTCF as follows. (4.4) can be generalized as

$$\rho_j(\eta_1+\eta) = z\mathrm{e}^{-\beta\widehat{W}(\eta_1)} j(x_1) \sum_{\xi\leq\eta} \int_{\Gamma_0} K(x_1;\xi+\gamma)\rho_j(\eta_1'+\eta+\gamma)\, \lambda_\sigma(d\gamma), \tag{4.5}$$

where we used in the expansion (4.2) the identity

$$\sum_{\xi\leq\gamma+\eta} K(x_1;\xi) = \sum_{\xi\leq\eta}\sum_{\varsigma\leq\gamma} K(x_1;\xi+\varsigma). \tag{4.6}$$

Inserting (4.5) into (3.26) (instead of ρ_j which contains η_1) and denoting $I_1 := J_1 \setminus \{1\}$, we have

$$\tilde{\rho}^T_{j;m}(\eta_1;\ldots;\eta_m) = z\mathrm{e}^{-\beta\widehat{W}(\eta_1)} j(x_1) \sum_{k=1}^m (-1)^{k-1}(k-1)!$$
$$\times \sum_{I_1\subset\{2,\ldots,m\}} \sideset{}{^*}\sum_{\{I_2,\ldots,I_k\}\subset\{2,\ldots,m\}\setminus I_1} \sum_{\xi\leq\sum_{i\in I_1}\eta_i} \int_{\Gamma_0} K(x_1;\xi+\gamma)\rho(\eta_1'+\sum_{i\in I_1}\eta_i+\gamma) \prod_{l=2}^k \rho(\sum_{i\in I_l}\eta_i)\, \lambda_\sigma(d\gamma).$$

Note that, in the second sum, the set I_1 can take on the value $I_1 = \emptyset$ in contrast to $I_2, \ldots, I_k$ in the 3-d sum. Changing the order of summations over indices I and over sets ξ, we may write

$$\widetilde{\rho}^T_{j;m}(\eta_1; \ldots; \eta_m) = z\mathrm{e}^{-\beta \widehat{W}(\eta_1)} j(x_1) \sum_{\xi \le \sum\limits_{i \in I_1} \eta_i} \int_{\Gamma_0} \lambda_\sigma(d\gamma)\, K(x_1; \xi + \gamma) \sum_{k=1}^{m - |I_0(\xi)|} (-1)^{k-1}(k-1)!$$

$$\times \sum_{I_1 \subset \{2,\ldots,m\}} \sum^{*}_{\{I_2,\ldots,I_k\} \subset \{2,\ldots,m\} \setminus (I_0(\xi) \cup I_1)} \rho(\eta_1^{'} + \sum_{i \in (I_0(\xi) \cup I_1)} \eta_i + \gamma) \prod_{l=2}^{k} \rho(\sum_{i \in I_l} \eta_i),$$

where we set $I_0(\xi) := \{i \ge 2 \,:\, \xi \cdot \eta_i \ne 0\}$. Setting now $\eta_{\{2,\ldots,m\} \setminus I} := (\eta_{i_2}; \ldots; \eta_{i_{m-|I|}})$ if $\{2, \ldots, m\} \setminus I = \{i_2, \ldots, i_{m-|I|}\}$, we arrive at

$$\widetilde{\rho}^T_{j;m}(\eta_1; \ldots; \eta_m) =$$
$$z\mathrm{e}^{-\beta \widehat{W}(\eta_1)} j(x_1) \sum_{\xi \le \sum\limits_{i=2}^{m} \eta_i} \int_{\Gamma} K(x_1; \xi + \gamma) \widetilde{\rho}^T_{m-|I_0(\xi)|}(\eta_1^{'} + \sum_{i \in I_0(\xi)} \eta_i + \gamma; \eta_{\{2,\ldots,m\} \setminus I_0(\xi)})\, \lambda_\sigma(d\gamma).$$

Here is the final rewriting for the key equation (recursion relation)

$$\widetilde{\rho}^T_{j;m}(\eta_1; \ldots; \eta_m) = z\mathrm{e}^{-\beta \widehat{W}(\eta_1)} j(x_1)$$
$$\times \sum_{I \subset \{2,\ldots,m\}} \sum^{*}_{\xi \le \sum\limits_{i \in I} \eta_i} \int_{\Gamma_0} K(x_1; \xi + \gamma) \widetilde{\rho}^T_{j;m-|I|}(\eta_1 - \mathbf{1}_{\{x_1\}} + \sum_{i \in I} \eta_i + \gamma; \eta_{\{2,\ldots,m\} \setminus I})\, \lambda_\sigma(d\gamma), \quad (4.7)$$

where the asterisk over the second sum means that for all $i \in I$, $\xi \cdot \eta_i \ne 0$. We emphasize that these equations hold provided that $|\eta_1| > 0$. They express the m-point j-PTCF $\widetilde{\rho}^T_{j;m}$ in terms of the r-point j-PTCF $\widetilde{\rho}^T_{j;r}$, $r \le m$. They therefore determine $\widetilde{\rho}^T_{j;m}$ uniquely if the operator

$$f \mapsto z\, \mathrm{e}^{-\beta \widehat{W}(\eta)} \chi_j(\eta)(1 - \delta_0(\eta)) \int_{\Gamma_0} K(x_1; \gamma) f(\gamma + \eta - \mathbf{1}_{\{x_1\}})\, \lambda_\sigma(d\gamma),$$

has $L^1(\Gamma_0, \lambda_\sigma)$-norm less than 1. Here, $\delta_0(\eta) := 1$ if $|\eta| = 0$, $\delta_0(\eta) := 0$ otherwise.

Remark 4.1 *We point out that* (4.7) *can be alternatively obtained by taking the derivatives of* (4.3) *with respect to* $\alpha_2, \ldots, \alpha_m$*, see* (3.24)–(3.25)*, and by using* (3.26) *along with* (4.6)*.*

4.2 Solution in the thermodynamic limit.

Due to the assumption $U(\gamma) = +\infty$ if $\gamma(x) > 1$ for some $x \in \mathbb{R}^d$, see (3.3), we can restrict ourselves to configurations such that $\gamma \le 1$. We then adopt set notation from now on and write γ for the set of points x with $\gamma(x) = 1$.

Following the strategy used in [22], we seek a solution of the equation (4.7) in the form

$$\widetilde{\rho}^T_{j;m}(\eta_1; \ldots; \eta_m) = \int_{\Gamma_0} \chi_j(\bigcup_{i=1}^{m} \eta_i \cup \gamma) T_m(\eta_1; \ldots; \eta_m \mid \gamma)\, \lambda_\sigma(d\gamma), \quad (4.8)$$

where $T_m(\eta_1; \ldots; \eta_m \mid \gamma)$, $m \ge 2$ and $\gamma \in \Gamma_0$ is a family of kernels such that

$$T_m(\eta_1; \ldots; \eta_m \mid \gamma) = 0 \;\text{ if } \gamma \cap \overline{\eta} \ne \emptyset, \quad \overline{\eta} := \bigcup_{i=1}^{m} \eta_i.$$

Inserting the expression (4.8) for $\widetilde{\rho}^T_{j;m}$ and $\widetilde{\rho}^T_{j;m-|I|}$ in both sides of (4.7) and then applying Lemma 2.2 (extended to the configuration space Γ_0), we arrive at the following recursion relations for the kernels

$T_m(\eta_1;\ldots;\eta_m \mid \gamma)$, owing to the arbitrariness of the function j,

$$T_m(\eta_1;\ldots;\eta_m \mid \gamma) = \\ z\mathrm{e}^{-\beta\widehat{W}(\eta_1)} \sum_{\xi\subset\gamma} \sum_{I\subset\{2,\ldots,m\}} \sum_{\eta\subset\overline{\eta}_I}^{*} K(x_1;\eta\cup\xi) T_{m-|I|}(\eta_1'\cup\overline{\eta}_I\cup\xi;\eta_{\{2,\ldots,m\}\setminus I} \mid \gamma\setminus\xi), \quad (4.9)$$

where $\eta_1' = \eta_1 \setminus \{x_1\}$ (set notation) and where we set $\overline{\eta}_I := \bigcup_{i\in I}\eta_i$. Subject to the initial conditions

$$T_1(\emptyset \mid \emptyset) = 1, \quad T_1(\emptyset \mid \gamma) = 0 \text{ if } \gamma \neq \emptyset, \quad (4.10)$$

and also, for all $m > 1$,

$$T_m(\eta_1;\ldots;\eta_m \mid \gamma) = 0 \text{ if } \gamma \neq \emptyset \text{ and } \eta_i = \emptyset \text{ for some } i = 1,\ldots,m; \quad (4.11)$$

the equation (4.9) has a unique solution due to its recursive structure. Indeed, these conditions follow from the fact that $\rho_j(\emptyset) = 1$ and $\widetilde{\rho}^{T}_{j;m}(\eta_1;\ldots;\eta_m) = 0$ if $\eta_i = \emptyset$ for some $i = 1,\ldots,m$, see Remark 3.4.

The main result of this section is a uniqueness result in the infinite-volume limit (i.e., $j \to 1$)

Theorem 4.2 *Assume that the interaction potential ϕ satisfies* (3.1) *and* (3.2)*. Given $m \in \mathbb{N}$, $m \geq 2$, there exists, for all $\beta > 0$, a unique solution of the equation* (4.7) *in the thermodynamic limit $j \to 1$, which can be written in the form*

$$\widetilde{\rho}^{T}_{m}(\eta_1;\ldots;\eta_m) = \int_{\Gamma_0} T_m(\eta_1;\ldots;\eta_m \mid \gamma)\,\lambda_\sigma(d\gamma), \quad (4.12)$$

where the family of kernels $|T_m(\eta_1;\ldots;\eta_m \mid \gamma)|$, $\gamma \in \Gamma_0$ is bounded above by a power series in the activity z (with integrable coefficients) which converges in the region

$$z\mathrm{e}^{2\beta B+2}\nu_1(\beta) < 1. \quad (4.13)$$

Here, $B \geq 0$ and $\nu_1(\beta) > 0$ are respectively defined in (3.5) *and* (3.6)*.*

The remaining of this paragraph is devoted to the proof of Theorem 4.2. For reader's convenience, the proofs of the intermediate results are placed in Sec. 4.3. Note that, in order to prove that (4.8)–(4.9) with the conditions (4.10)–(4.11) is a solution of the equation (4.7) as $j \to 1$, it is necessary to show that the kernels $T_m(\eta_1;\ldots;\eta_m \mid \gamma)$ are integrable functionals of the variable γ with respect to the measure λ_σ.

Following [22], given $h > 0$ and any bounded nonnegative even function $\nu : \mathbb{R}^d \to [0,+\infty)$, introduce a new family of kernels $Q_m(\eta_1;\ldots;\eta_m \mid \gamma)$, $m \geq 2$ and $\gamma \in \Gamma_0$ which are uniquely determined by the following system of recursion relations

$$Q_m(\eta_1;\ldots;\eta_m \mid \gamma) = h \sum_{\xi\subset\gamma} \sum_{I\subset\{2,\ldots,m\}} \sum_{\eta\subset\overline{\eta}_I}^{*} K_\nu(x_1;\eta\cup\xi) Q_{m-|I|}(\eta_1'\cup\overline{\eta}_I\cup\xi;\eta_{\{2,\ldots,m\}\setminus I} \mid \gamma\setminus\xi), \quad (4.14)$$

with initial conditions

$$Q_1(\emptyset \mid \emptyset) = 1, \quad Q_1(\emptyset \mid \gamma) = 0 \text{ if } \gamma \neq \emptyset, \quad (4.15)$$

and also, for all $m > 1$,

$$Q_m(\eta_1;\ldots;\eta_m \mid \gamma) = 0 \text{ if } \gamma \neq \emptyset \text{ and } \eta_i = \emptyset \text{ for some } i = 1,\ldots,m, \quad (4.16)$$

where

$$K_\nu(x_1;\xi) := \begin{cases} 1, & \text{if } \xi = \emptyset, \\ \prod_{x\in\xi} \nu(x_1 - x), & \text{if } |\xi| \geq 1. \end{cases} \quad (4.17)$$

Since, by using assumption (3.5), we have from the expression (4.9)

$$|T_m(\eta_1;\ldots;\eta_m \mid \gamma)| \leq z\mathrm{e}^{2\beta B} \sum_{\xi\subset\gamma} \sum_{I\subset\{2,\ldots,m\}} \sum_{\eta\subset\overline{\eta}_I}^{*} |K(x_1;\eta\cup\xi)||T_{m-|I|}(\eta_1'\cup\overline{\eta}_I\cup\xi;\eta_{\{2,\ldots,m\}\setminus I} \mid \gamma\setminus\xi)|,$$

the following result, which can be proven by induction, is straightforward

Lemma 4.3 *Assume that the interaction potential ϕ satisfies* (3.1)–(3.2)*. Given $\beta > 0$ and $z > 0$, set*

$$ze^{2\beta B} = h \quad \textit{and} \quad |e^{-\beta\phi(|x-y|)} - 1| = \nu(x-y), \tag{4.18}$$

where $B \geq 0$ is defined in (3.5)*. Then, given $m \in \mathbb{N}$, $m \geq 2$ and $\gamma \in \Gamma_0$, the following holds,*

$$|T_m(\eta_1; \ldots; \eta_m \mid \gamma)| \leq Q_m(\eta_1; \ldots; \eta_m \mid \gamma). \tag{4.19}$$

The solution $Q_m(\eta_1; \ldots; \eta_m \mid \gamma)$ of the equation (4.14) with conditions (4.15)–(4.17) can be written with the help of *forest graphs*. For each set of clusters $\{\eta_1; \ldots; \eta_m\}$ with $\eta_j \in (\Gamma_0 \backslash \emptyset)$ and each configuration $\gamma \in \Gamma_0$, we define the set of forest graphs $\mathfrak{S}(\eta_1; \ldots; \eta_m \mid \gamma)$ in the following way. The connected components of the graphs $\widetilde{f} \in \mathfrak{S}(\eta_1; \ldots; \eta_m \mid \gamma)$ are tree graphs with vertices given by points of $\bigcup_{i=1}^m \eta_i \cup \gamma$, and such that there are no lines (or edges) connecting vertices of the same cluster η_i (for $i = 1, \ldots, m$). Each tree contains a point of $\bigcup_{i=1}^m \eta_i$ and, if i_0 is the lowest index such that η_{i_0} contains a point of the tree, then this point is unique (the *root* of the tree). Moreover, for every other vertex z of the tree there is a path $z_1, \ldots, z_k$ such that $z_k = z$, and there is an edge between the root x_0 and z_1 and between each pair z_p and z_{p+1}, and such that if $z_p \in \eta_{i_p}$ then, if $z_{p+1} \in \eta_{i_{p+1}}$ then z_p is the only point in η_{i_p} connected to a point in $\eta_{i_{p+1}}$ by a line *in the forest*, whereas if $z_{p+1} \in \gamma$ then z_p is the only point in η_{i_p} to which it is connected by a line in the forest. Note that a single point $x \in \bigcup_{i=1}^m \eta_i$ is also a tree with analytic contribution h. Finally, if all points of the configurations η_i (for every $i = 1, \ldots, m$) are combined into one single vertex, then the forest graph $\widetilde{f} \in \mathfrak{S}(\eta_1; \ldots; \eta_m \mid \gamma)$ reduces to a connected tree graph with $m + n$ vertices, where $n = |\gamma|$.

With the above notation, we now establish the following lemma

Lemma 4.4 *The solution of the equation* (4.14) *with conditions* (4.15)–(4.17) *can be written as*

$$Q_m(\eta_1; \ldots; \eta_m \mid \gamma) = \sum_{\widetilde{f} \in \mathfrak{S}(\eta_1; \ldots; \eta_m \mid \gamma)} G_\nu(\widetilde{f}), \tag{4.20}$$

where the analytic contribution of a forest graph $\widetilde{f} \in \mathfrak{S}(\eta_1; \ldots; \eta_m \mid \gamma)$, denoted by $G_\nu(\widetilde{f})$, is given by

$$G_\nu(\widetilde{f}) = G_\nu(\widetilde{f}; \eta_1; \ldots; \eta_m \mid \gamma) = h^{l+|\gamma|} \prod_{(x,y) \in E(\widetilde{f})} \nu(x-y), \tag{4.21}$$

where $E(\widetilde{f})$ denotes the set of the edges of $\widetilde{f}$, and where,

$$l := \sum_{i=1}^m l_i \ \ \textit{with} \ l_i := |\eta_i|, \quad i = 1, \ldots, m. \tag{4.22}$$

Individual analytic contributions are easily estimated

Lemma 4.5 *Set*

$$\nu_0 := \max_{x \in \mathbb{R}^d} \nu(x) < +\infty, \tag{4.23}$$

$$\nu_1 := \int_{\mathbb{R}^d} \nu(x)\, dx < +\infty. \tag{4.24}$$

Then, given a forest graph $\widetilde{f} \in \mathfrak{S}(\eta_1; \ldots; \eta_m \mid \gamma)$ with $|\gamma| = n \in \mathbb{N}$,

$$\int_{\mathbb{R}^{dn}} G_\nu(\widetilde{f}; \eta_1; \ldots; \eta_m \mid \{y_1, \ldots, y_n\})\, dy_1 \cdots dy_n \leq h^{l+n} \nu_0^{|E_{\overline{\eta}}(\widetilde{f})|} \nu_1^n, \tag{4.25}$$

where l is defined in (4.22)*, and*

$$|E_{\overline{\eta}}(\widetilde{f})| \leq l - l_1, \quad \overline{\eta} := \bigcup_{i=1}^m \eta_i,$$

stands for the number of edges in which one or two ends belong to the set $\bigcup_{i=1}^m \eta_i$.

It remains to estimate the number of forest graphs at fixed configurations $\bigcup_{i=1}^m \eta_i \cup \gamma$. We denote this number by $N_n^{(m)}(l_1, \dots, l_m)$, $l_i := |\eta_i|$ and prove the following combinatoric lemma.

Lemma 4.6 *Let $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, $m \geq 2$. Set $L_i := 2^{l_i} - 1$ for $i = 2, \dots, m$. Then,*

$$N_n^{(m)}(l_1; \dots; l_m) = l_1 (\prod_{i=2}^m L_i) \left(\sum_{i=1}^m l_i + n \right)^{m+n-2} . \tag{4.26}$$

Remark 4.7 *We point out that* (4.26) *is a generalization of (well-known) Cayley's formula for the number of tree graphs with $n \geq 2$ vertices*

$$K_n = n^{n-2},$$

for the case of forest graphs of a system of m clusters $(\eta_i)_{i=1}^m$ and n single vertices with $l_i = |\eta_i|$, $n = |\gamma|$.

We finally turn to

Proof of Theorem 4.2. Assume that the interaction potential ϕ satisfies (3.1)–(3.2). Let $\beta > 0$ and $z > 0$ satisfying (4.13). In view of (4.18) and (4.23)–(4.24), set

$$\nu_0(\beta) := \max_{x \in \mathbb{R}^d} \nu_\beta(x) = \max_{x \in \mathbb{R}^d} |\mathrm{e}^{-\beta\phi(|x|)} - 1| < +\infty.$$

Note that $\nu_1(\beta) > 0$ is defined in (3.6). From (4.19) along with (4.20), we have,

$$|T_m(\eta_1; \dots; \eta_m \mid \gamma)| \leq Q_m(\eta_1; \dots; \eta_m \mid \gamma) = \sum_{\widetilde{f} \in \mathfrak{S}(\eta_1; \dots; \eta_m | \gamma)} G_\nu(\widetilde{f}),$$

where $G_\nu(\widetilde{f})$ is a monomial in z of order $l + n$. Inserting now the bounds (4.25) and (4.26), we get,

$$|\widetilde{\rho}_m^T(\eta_1; \dots; \eta_m)| \leq \left(2z\mathrm{e}^{2\beta B}\right)^l (\nu_0(\beta))^{l-l_1} \sum_{n=0}^\infty \frac{(l+n)^{m+n-2}}{n!} \left(z\mathrm{e}^{2\beta B} \nu_1(\beta)\right)^n ,$$

where we replaced ν_k by $\nu_k(\beta)$, $k = 0, 1$. Applying Stirling formula, i.e., $n! > n^n \mathrm{e}^{-n}\sqrt{2\pi n}$, we have,

$$|\widetilde{\rho}_m^T(\eta_1; \dots; \eta_m)| \leq \left(2z\mathrm{e}^{2\beta B+1}\right)^l (\nu_0(\beta))^{l-l_1} \, l^{m-2} \sum_{n=0}^\infty \left(z\mathrm{e}^{2\beta B+2} \nu_1(\beta)\right)^n .$$

Note that, we also used the bound $(1 + \frac{n}{l})^{m-2} \leq \mathrm{e}^n$ which follows from $l \geq m - 2$. ■

We conclude this section by

Remark 4.8 *Analogous to* (4.20)*, there is an analytic expression for the kernels $T_m(\eta_1; \dots; \eta_m \mid \gamma)$ in terms of forest graphs*

$$T_m(\eta_1; \dots; \eta_m \mid \gamma) = \sum_{\widetilde{f} \in \mathfrak{S}(\eta_1; \dots; \eta_m | \gamma)} G_C(\widetilde{f}), \tag{4.27}$$

where the contribution C_{xy} for an edge of $G_C(\widetilde{f})$ connecting vertices x and y is given by (3.18)*, and where the analytic expression for $G_C(\widetilde{f})$ has the more complicated form*

$$G_C(\widetilde{f}) = G_C(\widetilde{f}; \eta_1; \dots; \eta_m \mid \gamma) = z^{l+n} \prod_{(x,y) \in E(\widetilde{f})} C_{xy} \prod_{(x,y) \in S(\widetilde{f})} \mathrm{e}^{-\beta\phi(|x-y|)}, \tag{4.28}$$

where $S(\widetilde{f})$ denotes the set of pairs of points of the set $\bigcup_{i=1}^m \eta_i \cup \gamma$ for which there are no edges in the graph forest $\widetilde{f}$.

Remark 4.9 *Obviously, ordinary truncated (connected) correlation functions are a special case obtained by taking $\eta_1 = \{x_1\}, \dots, \eta_m = \{x_m\}$ in* (4.27) *and* (4.28)*. In this case, each term of the expansion is the sum of the contributions of the connected Cayley tree-graphs, and the expansion itself coincides with that obtained by O. Penrose in [27].*

4.3 Proof of Lemmas 4.4, 4.5 and 4.6.

Proof of Lemma 4.4. The proof is done by induction on $n = |\gamma|$. Consider first the case $n = 0$, so $\gamma = \emptyset$. The equation (4.14) reduces then to

$$Q_m(\eta_1; \dots; \eta_m \mid \emptyset) = h \sum_{I \subset \{2,\dots,m\}} \sum_{\eta \subset \overline{\eta}_I}^{*} K_\nu(x_1; \eta) Q_{m-|I|}(\eta_1' \cup \overline{\eta}_I; \eta_{\{2,\dots,m\}\setminus I} \mid \emptyset). \tag{4.29}$$

In particular, $Q_1(\eta_1 \mid \emptyset) = hQ_1(\eta_1' \mid \emptyset)$, so that $Q_1(\eta_1 \mid \emptyset) = h^{|\eta_1|}$. This agrees with (4.20) since the only allowed tree consists of individual points $x \in \eta_1$. We now do induction on m and $l_1 = |\eta_1|$. For $m = 1$, we already have that $Q_1(\eta_1 \mid \emptyset) = h^{l_1}$. Assuming that $Q_1 \dots, Q_{m-1}$ are given by the sum of forest contributions when $\gamma = \emptyset$, the terms in (4.29) correspond to the construction of a forest on $\bigcup_{i=1}^m \eta_i$ as follows. The point x_1 is connected to a set of points η outside η_1. If I is the set of indices such that $\eta \cap \eta_i \neq \emptyset$, then in $Q_{m-|I|}(\eta_1' \cup \overline{\eta}_I; \eta_{I^c} \mid \emptyset)$ there are no more connections within $\eta_1' \cup \overline{\eta}_I$, i.e., between any other points of η_1 and points of $\overline{\eta}_I$ or between two points of $\overline{\eta}_I$. In $Q_{m-|I|}$ either $m - |I| < m$ or $I = \emptyset$, in which case the first subset is η_1' and $|\eta_1'| < |\eta_1|$. Therefore, by the induction hypothesis, its contributions are forest graphs with vertices in $\eta_1' \cup \bigcup_{i=2}^m \eta_i$ such that each tree contains at most one point of $\eta_1' \cup \overline{\eta}_I$. This means that when the connections with x_1 are added, the resulting graph still consists of separate trees. Denote the resulting forest graph on $\bigcup_{i=1}^m \eta_i$ by $\widetilde{f}$. If $x \neq x_1$ is a vertex in $\widetilde{f}$, then by the induction hypothesis, there is a sequence of points $z_0 \in \eta_1' \cup \bigcup_{i=2}^m \eta_i$, $z_1, \dots, z_k \in \overline{\eta}_{I^c}$ such that $z_k = x$ and if $z_p \in \eta_{i_p}$ $(p = 0, \dots, k)$ then z_p is the unique point in η_{i_p} connected to a point in $\eta_{i_{p+1}}$ by a line in $\widetilde{f}$ (note that $z_0 = x$ if $x \in \eta_1' \cup \overline{\eta}_I$.) Now, if $z_0 \in \eta_1'$ or $z_0 \in \eta_i \setminus \eta$, then it is the root of a tree in $\widetilde{f}$. If $z_0 \in \eta_i \cup \eta$ then x_1 is the root of the tree containing x and there is no other point $x' \in \eta_1'$ connected to a point in η_i by a line in $\widetilde{f}$. Collapsing the points of $\{x_1\} \cup \eta$ to a single point, the forest reduces to a forest on $\eta_1' \cup \overline{\eta}_I$ and $\bigcup_{i \in \{2,\dots,m\}\setminus I} \eta_i$ because there are no more edges in $\widetilde{f}$ between points of $\eta_1 \cup \overline{\eta}_I$. The resulting forest is precisely one of the contributions to $Q_{m-|I|}(\eta_1' \cup \overline{\eta}_I; \eta_{I^c} \mid \emptyset)$. If each η_i is reduced to a point, the resulting graph is connected by induction except possibly in the case that $\eta_1 = \{x_1\}$ and $\eta = \emptyset$. But in that case, if $m > 1$, the contribution $Q_m(\eta_1'; \eta_2; \dots; \eta_m | \emptyset) = 0$ since $\eta_1' = \emptyset$. The powers of h are obviously correct.
It remains to do induction on n. The term $\xi = \emptyset$ gives the contribution

$$h \sum_{I \subset \{2,\dots,m\}} \sum_{\eta \subset \overline{\eta}_I}^{*} K_\nu(x_1; \eta) Q_{m-|I|}(\eta_1' \cup \overline{\eta}_I; \eta_{\{2,\dots,m-k\}\setminus I} \mid \gamma).$$

This is similar to the case $\gamma = \emptyset$. It corresponds to the case where x_1 is only connected to points in $\bigcup_{i=2}^m \eta_i$, and the remaining tree after collapsing the points $\{x_1\} \cup \overline{\eta}_I$ gives the stated contribution by induction, since either $m - |I| < m$ or $|\eta_1'| < |\eta_1|$. Once again the contribution of $I = \emptyset$ is zero if $\eta_1 = \{x_1\}$. The other terms are more complicated. Now, x_1 is connected to a set of points $\xi \subset \gamma$ as well as a set of points $\eta \subset \bigcup_{i=2}^m \eta_i$. Collecting the points of $\{x_1\} \cup \xi \cup \eta$ into a single vertex, the corresponding forest is just the contribution to $Q_{m-|I|}(\eta_1' \cup \overline{\eta}_I \cup \xi; \eta_{I^c} \mid \gamma \setminus \xi)$ by induction since $|\gamma \setminus \xi| < n$. For $y \in \xi$, there are no more lines between other points of η_1 and y. Also, there are no more lines connecting $x \in \overline{\eta}_I$ to another point of η_1. It remains to show that upon collapsing the points of each η_i to a single vertex, the resulting graph is a connected tree. This is more intricate. We first prove connectedness. Suppose there is an index set $J \subset \{1, \dots, m\}$ (with $1 \in J$) and a subset $\zeta \subset \gamma$ such that $J \neq \{1, \dots, m\}$ or $\zeta \neq \gamma$ and there are no lines between points of $\overline{\eta}_J \cup \zeta$ and points in the complement. Clearly, $I \subset J$ since the points of η are connected to x_1 (due to the factor $K_\nu(x_1; \xi \cup \eta)$) and $\eta \cap \eta_i \neq \emptyset$ for $i \in I$. Also $\xi \subset \zeta$ for the same reason. By the induction hypothesis for $Q_{m-|I|}(\eta_1' \cup \overline{\eta}_I \cup \xi; \eta_{I^c} \mid \gamma \setminus \xi)$, there are no sets $J' \supset I$ and $\zeta' \subset \gamma \setminus \xi$ without external lines, other than the trivial $J' = \emptyset$ and $\zeta' = \emptyset$ or $J' = I \cup I^c$ and $\zeta' = \gamma \setminus \xi$. Therefore $J \supset \{2, \dots, m\}$ and $\gamma \setminus \xi \subset \zeta$, or $J \cap \{2, \dots, m\} = \emptyset$ and $\zeta \subset \gamma \setminus \xi$. In the first case $J = \{1, \dots, m\}$ and $\zeta = \gamma$, which contradicts the initial assumption. In the second case, $J = \{1\}$ and $\xi = \emptyset$, and since $I \subset J$, $I = \emptyset$. The corresponding contribution equals zero as above, since then $\eta_1' \cup \overline{\eta}_I \cup \xi = \emptyset$. To see that the resulting graph is a tree, note that in any contributing forest to $Q_{m-|I|}(\eta_1' \cup \xi \cup \overline{\eta}_I; \eta_{I^c} \mid \gamma \setminus \xi)$ there is just one line between a point of $\eta_1' \cup \xi \cup \overline{\eta}_I$ and a tree on $\eta_{I^c} \cup \gamma \setminus \xi$. The factor $K_\nu(x_1; \xi \cup \eta)$ gives lines between x_1 and the points of $\xi \cup \eta$, and therefore to only one point of this tree. ■

Proof of Lemma 4.5. We only give the main arguments. If y_i is an end vertex of a tree in the forest $\widetilde{f}$, then a contribution involving the factor ν_1 arises. The same holds if, from y_i outwards, there are only vertices y_k since we can integrate them successively. In the case when y_i lies between the points x_i and x_k in $\bigcup_{i=1}^{m} \eta_i$, we first use the inequality $\nu(y_i - x_k) \leq \nu_0$. ■

Proof of Lemma 4.6. From (4.14), it can be seen that $N_n^{(m)}(l_1; \dots; l_m)$ satisfies the recurrent relations

$$N_n^{(m)}(l_1; \dots; l_m) = \sum_{k=0}^{n} \binom{n}{k} \sum_{I \subset \{2,\dots,m\}} L_I N_{n-k}^{(m-|I|)}(l_1 + l_I + k - 1; l_{i_2}; \dots; l_{i_{m-|I|}}), \tag{4.30}$$

where we denote $L_I := \prod_{i \in I} L_i$, $l_I := \sum_{i \in I} l_i$ (with the convention $l_\emptyset := 0$) and $\{i_2, \dots, i_{m-|I|}\} := \{2, \dots, m\} \setminus I$. Hereafter, we set $l := \sum_{i=1}^{m} l_i$. Let us introduce new numbers $\tilde{N}_n^{(m)}(l_1; \dots; l_m)$ in such a way that

$$N_n^{(m)}(l_1; \dots; l_m) = (\prod_{i=2}^{m} L_i) \tilde{N}_n^{(m)}(l_1; \dots; l_m).$$

Then, the recurrent relations (4.30) can be rewritten in the following way,

$$\tilde{N}_n^{(m)}(l_1; \dots; l_m) = \sum_{k=0}^{n} \binom{n}{k} \sum_{I \subset \{2,\dots,m\}} \tilde{N}_{n-k}^{(m-|I|)}(l_1 + l_I + k - 1; l_{i_2}; \dots; l_{i_{m-|I|}}). \tag{4.31}$$

We now prove that, given the initial condition $\tilde{N}_0^{(1)}(l_1) = 1$,

$$\tilde{N}_n^{(m)}(l_1; \dots; l_m) = l_1 (\sum_{i=1}^{m} l_i + n)^{m+n-2} \tag{4.32}$$

is the solution of the recurrent relations (4.31). Inserting the identity

$$\tilde{N}_{n-k}^{(m-|I|)}(l_1 + l_I + k - 1; l_{i_2}; \dots; l_{i_{m-|I|}}) = (l_1 + l_I + k - 1)(l + n - 1)^{m+n-k-|I|-2},$$

in the right-hand side of (4.31), we obtain,

$$\tilde{N}_n^{(m)}(l_1; \dots; l_m) = \sum_{i=1}^{3} M_i,$$

where,

$$M_1 := \sum_{k=0}^{n} \binom{n}{k} (l + n - 1)^{n-k-1} \sum_{I \subset \{2,\dots,m\}} l_1 (l + n - 1)^{m-|I|-1},$$

$$M_2 := \sum_{k=0}^{n} \binom{n}{k} (l + n - 1)^{n-k-1} \sum_{I \subset \{2,\dots,m\}} l_I (l + n - 1)^{m-|I|-1},$$

$$M_3 := \sum_{k=0}^{n} \binom{n}{k} (l + n - 1)^{n-k-1} \sum_{I \subset \{2,\dots,m\}} (k - 1)(l + n - 1)^{m-|I|-1}.$$

Now, in M_2 we first sum over sets I with $|I| = p$ using

$$\sum_{\substack{I \subset \{2,\dots,m\} \\ |I|=p}} l_I = \binom{m-2}{p-1} \sum_{i=2}^{m} l_i = \binom{m-2}{p-1} (l - l_1).$$

In the other two sums, this summation is easy, and we obtain

$$\begin{aligned}
M_1 &= l_1 \sum_{k=0}^{n} \binom{n}{k} (l+n-1)^{n-k-1} \sum_{p=0}^{m-1} \binom{m-1}{p} (l+n-1)^{m-p-1} \\
&= l_1 \sum_{k=0}^{n} \binom{n}{k} (l+n-1)^{n-k-1} (l+n)^{m-1} = l_1 (l+n-1)^{-1} (l+n)^{m+n-1}, \\
M_2 &= (l-l_1) \sum_{k=0}^{n} \binom{n}{k} (l+n-1)^{n-k-1} \sum_{p=1}^{m-1} \binom{m-2}{p-1} (l+n-1)^{m-p-1} \\
&= (l-l_1)(l+n-1)^{-1}(l+n)^{m+n-2}, \\
M_3 &= \sum_{k=0}^{n} \binom{n}{k} (l+n-1)^{n-k-1} \sum_{p=0}^{m-1} \binom{m-1}{p} (k-1)(l+n-1)^{m-p-1} \\
&= \sum_{k=0}^{n} \binom{n}{k} (k-1)(l+n-1)^{n-k-1} (l+n)^{m-1} = -l(l+n-1)^{-1}(l+n)^{m+n-2},
\end{aligned}$$

where we used the identity

$$\sum_{k=0}^{n} \binom{n}{k} (k-1)(l+n-1)^{n-k-1} \ = \ -l(l+n-1)^{-1}(l+n)^{n-1}.$$

We conclude that $\sum_{i=1}^{3} M_i = l_1(l+n)^{m+n-2}$ which completes the induction. This proves (4.32). ■

5 Strong decay properties for PTCF.

Theorem 4.2 states the existence of a unique solution to the equation (4.7) in the form of convergent expansions, see (4.12) with Remark 4.8 and also (4.19) with (4.20)–(4.21). The most important property of the PTCF is their decay as the distances between the clusters increases, i.e., $\operatorname{dist}(\eta_i, \eta_j) \to +\infty$, $i \neq j$.

5.1 Polynomial decay for PTCF.

We start by formulating the main result of Sec. 5

Theorem 5.1 *Suppose that the interaction potential ϕ satisfies* (3.5) *and* (3.6)*. Assume in addition that there exists $\alpha > d$ and, for all $\beta > 0$, there exists a constant $C(\beta) > 0$ such that*

$$\nu_\beta(x) := |\mathrm{e}^{-\beta\phi(x)} - 1| \leq C(\beta)\overline{\nu}(x),$$

with

$$\overline{\nu}(x) := \frac{1}{1+|x|^\alpha}, \quad x \in \mathbb{R}^d. \tag{5.1}$$

Then, provided that,

$$z\mathrm{e}^{2\beta B}[\nu_1(\beta)\mathrm{e} + \overline{\nu}_1(\beta)(\mathrm{e} + 2^{1+\alpha})] < 1,$$

where $B \geq 0$ is defined in (3.5)*, $\nu_1(\beta) > 0$ in* (3.6) *and*

$$\overline{\nu}_1(\beta) := C(\beta)\overline{\nu}_1, \quad \overline{\nu}_1 := \int_{\mathbb{R}^d} \overline{\nu}(x)\, dx < +\infty,$$

there exist, given $m \in \mathbb{N}$ with $m \geq 2$, constants $A_{m,\sigma} = A_{m,\sigma}(\beta, z, \alpha) > 0$, $1 \leq \sigma \leq m$ such that the PTCF in (4.12) *admit the following bounds*

$$|\tilde{\rho}_m^T(\eta_1; \ldots; \eta_m)| \leq \sum_{\sigma=1}^{m} A_{m,\sigma} \max_{T_m \in \mathcal{T}_m} \overline{\nu}_{T_m},$$

where $\mathcal{T}_m$ denotes the set of trees on m points, and

$$\overline{\nu}_{T_m} := \prod_{(i,j)\in T_m} \max_{x_i \in \eta_i; x_j \in \eta_j} \overline{\nu}(x_i - x_j).$$

Remark 5.2 *Explicit upper bounds for the constants $A_{m,\sigma}$ are derived in the proof. Setting*

$$z\mathrm{e}^{2\beta B} = h, \quad \nu_1(\beta) = \nu_1, \quad C(\beta) = C,$$

$A_{2,\sigma}$ with $\sigma = 1, 2$ are given in (5.12)*, $A_{3,\sigma}$ with $\sigma = 1, 2, 3$ are given in* (5.19)*,* (5.20) *and* (5.21)*, and for any $m \geq 4$, $A_{m,1}$, $A_{m,2}$ and $A_{m,\sigma}$ with $3 \leq \sigma \leq m$ are given in* (5.23)*,* (5.25) *and* (5.34) *respectively. To derive these upper bounds, we use the combinatoric identities* (5.33)*.*

The rest of Sec. 5 is devoted to the proof of Theorem 5.1. It is organized as follows. We first establish two technical results, see Lemma 5.3 and Proposition 5.1 below. Subsequently, we prove Theorem 5.1 in the case $m = 2$, $m = 3$ and the general case $m \geq 4$ in Sec. 5.2, 5.3 and 5.4 respectively.

We point out that, from (4.12) with (4.9) (and the conditions (4.10)–(4.11)), it is sufficient, by virtue of Lemma 4.3, to work with the family of kernels in (4.14) (with the conditions (4.15)–(4.17)).

Consider for instance forest graphs $\widetilde{f} \in \mathfrak{S}(\eta_1; \ldots; \eta_m \mid \{y_1\})$. Restricting the diagram to $\bigcup_{i=1}^m \eta_i$, one obtains a forest on $\bigcup_{i=1}^m \eta_i$ of which some trees are connected by an edge in $\widetilde{f}$ to y_1. If there is just one such edge, the corresponding contribution is obtained from that of the restricted forest by multiplying by $\nu(x_j - y_1)$ if x_j is the vertex attached to y_1. In general, one has to multiply by a factor $\prod_{r=1}^p \nu(x_{j_r} - y_1)$. In the former case, integration with respect to the variable y_1 simply multiplies the contribution of the diagram from $\mathfrak{S}(\eta_1; \ldots; \eta_m \mid \emptyset)$ by the factor ν_1, see (4.24). In the general case, we need to consider integrals of the form

$$\int_{\mathbb{R}^d} \prod_{r=1}^{p} \nu(x_r - y)\, dy. \tag{5.2}$$

In case that the kernel ν has a polynomial decay, terms of type (5.2) can be easily estimated

Lemma 5.3 *Let $\overline{\nu}$ be the kernel in* (5.1) *with $\alpha > d$. Then, for all $p \in \mathbb{N}$, $p \geq 2$ and $x_1, \ldots, x_p \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \prod_{r=1}^{p} \overline{\nu}(x_r - y)\, dy \leq 2^{\alpha(p-1)} \overline{\nu}_1 \sum_{r=1}^{p} \prod_{\substack{k=1 \\ k \neq r}}^{p} \overline{\nu}(x_k - x_r). \tag{5.3}$$

Proof. We subdivide the integral with respect to y into domains where $|y - x_r| < \max_{k \neq r} |y - x_k|$. Then, $|x_k - y| > \frac{1}{2}|x_k - x_r|$ and the inequality (5.3) easily follows from

$$\frac{1}{|x_k - y|^\alpha + 1} < \frac{1}{(\frac{1}{2}|x_k - x_r|)^\alpha + 1} < \frac{2^\alpha}{|x_k - x_r|^\alpha + 1}. \qquad \blacksquare$$

To count the possible diagrams, we will first isolate the parts of the diagram consisting of trees with vertices in γ except possibly one endpoint. This can be done as follows. Define

$$\begin{aligned}
&Q_m(\eta_1; \ldots; \eta_m \mid 0) := Q_m(\eta_1; \ldots; \eta_m \mid \emptyset), \\
&Q_m(\eta_1; \ldots; \eta_m \mid n) := \int_{\mathbb{R}^{dn}} Q_m(\eta_1; \ldots; \eta_m \mid \{y_1, \ldots, y_n\})\, dy_1 \cdots dy_n, \quad n \in \mathbb{N},
\end{aligned} \tag{5.4}$$

where the family of kernels $Q_m(\eta_1; \ldots; \eta_m \mid \gamma)$, $m \geq 2$ and $\gamma \in \Gamma_0$ is given in (4.14) with the conditions (4.15)–(4.17). It then satisfies the following recursion relation

$$\begin{aligned}
Q_m(\eta_1; \ldots; \eta_m \mid n) = h &\sum_{I \subset \{2, \ldots, m\}} K^{(0)}(x_1; \overline{\eta}_I) \\
\times &\sum_{k=0}^{n} \binom{n}{k} \int_{\mathbb{R}^{dk}} \prod_{j=1}^{k} K_\nu(x_1; y_j) Q_{m-|I|}(\eta_1' \cup \overline{\eta}_I \cup \{y_1, \ldots, y_k\}; \eta_{\{2,\ldots,m\} \setminus I} \mid n-k)\, dy_1 \cdots dy_k,
\end{aligned} \tag{5.5}$$

where it is understood that the term $k = 0$ in the sum reduces to $Q_{m-|I|}(\eta_1' \cup \overline{\eta}_I; \eta_{\{2,\ldots,m\} \setminus I} \mid n)$, and

$$\begin{aligned}
&K^{(0)}(x_i; \emptyset) := 1, \quad i \in \{1, \ldots, m\}, \\
&K^{(0)}(x_i; \overline{\eta}_I) := \sum_{\eta \subset \overline{\eta}_I}^{*} K_\nu(x_i; \eta) = \sum_{\eta \subset \overline{\eta}_I}^{*} \prod_{x \in \eta} \nu(x_i - x), \quad I \subset \{1, \ldots, m\} \setminus \{i\}.
\end{aligned} \tag{5.6}$$

We then establish

Proposition 5.1 *Given $n \in \mathbb{N}_0$, the solution of the recursion relation* (5.5) *can be expressed as*

$$Q_m(\eta_1; \ldots; \eta_m \mid n) = h^l \sum_{k=0}^{n} \binom{n}{k} k! N^{(1)}_{n-k}(l+k)(h\nu_1)^{n-k} \widetilde{Q}_m(\eta_1; \ldots; \eta_m \mid k),$$

where l is defined in (4.22), *ν_1 in* (4.24), *$N^{(1)}_{k'}$ with $0 \le k' \le n$ is given by* (4.26), *and $\widetilde{Q}_m(\eta_1; \ldots; \eta_m \mid k)$ consists of the contributions from all forest graphs in $\mathfrak{S}(\eta_1; \ldots; \eta_m \mid \{y_1, \ldots, y_k\})$ in which all vertices of y_i of γ are connected to at least two other vertices.*

Proof. This can be proved inductively from the formula (5.5). However, it is also easily understood graphically as follows. Given a forest graph in $\mathfrak{S}(\eta_1; \ldots; \eta_m \mid \{y_1, \ldots, y_n\})$, consider the points of $\gamma = \{y_1, \ldots, y_n\}$ connected to only one other vertex (endpoints). These are parts of trees on γ with a single base point either in γ or in $\bigcup_{i=1}^m \eta_i$. Starting at the endpoints, the corresponding points y_i can easily be integrated, yielding factors $h\nu_1$. In the remaining graph, each point of γ is connected to at least two other vertices. We denote the contribution of this graph by $\widetilde{Q}_m(\eta_1; \ldots; \eta_m \mid k)$, where k is the number of remaining vertices in γ. Conversely, given a forest graph in $\mathfrak{S}(\eta_1; \ldots; \eta_m \mid \{y_1, \ldots, y_k\})$ in which each point y_i $(i = 1, \ldots, k)$ is connected to at least two other vertices, we obtain the contribution from graphs in $\mathfrak{S}(\eta_1; \ldots; \eta_m \mid \{y_1, \ldots, y_n\})$ with $n \ge k$, containing this graph and such that all other points $y_{k+1}, \ldots, y_n$ are in trees with a single base point, by counting the number of possibilities of attaching trees to the given tree with total number of vertices equal to $n - k$. But this number is given precisely by

$$\binom{n}{k} k! N^{(1)}_{n-k}(l+k)(h\nu_1)^{n-k}.$$

Indeed, we can choose which of the total of n points belongs to the original graph in $\binom{n}{k}$ ways and order them in $k!$ ways. The number of ways of forming trees out of the remaining $n - k$ points is then given by $N^{(1)}_{n-k}(l+k)$, because for this purpose we can consider all points of the original graph as belonging to a single cluster as they cannot be connected further to each other. There are obviously $l + k$ such points to be connected to a further $n - k$ external points. By Lemma 4.6, this can be done in $N^{(1)}_{n-k}(l+k)$ ways. ■

5.2 The case $m = 2$.

There are two possibilities: either there is at least one line between η_1 and η_2 in the forest, or there is none. In the first case, the restriction of the forest to γ splits into separate trees, each of which is connected to a single point of either η_1 or η_2. In the second case, the restriction to γ also splits into separate trees, but one of these is connected to a single point of η_1 as well as one or more points of η_2. The others are again connected to a single point of either η_1 or η_2. The trees connected to a single point are easily integrated out, giving rise to factors ν_1. If there is a tree connecting η_1 and η_2 then there is one point y_1 of that tree in γ connected to a point of η_1 and one point $y_2 \in \gamma$ connected to one or more points of η_2 (y_1 can be equal to y_2). In that case, there is a unique path in the tree connecting y_1 to y_2. The remaining part of the tree consists of individual trees connected to single points of this path (or points of $\eta_1 \cup \eta_2$). These can be integrated out giving factors ν_1 as before. In terms of Proposition 5.1,

$$Q_2(\eta_1; \eta_2 \mid n) = h^l \sum_{k=0}^{n} \binom{n}{k} k! N^{(1)}_{n-k}(l+k)(h\nu_1)^{n-k} \widetilde{Q}_2(\eta_1; \eta_2 \mid k), \quad n \in \mathbb{N}_0, \tag{5.7}$$

with

$$\widetilde{Q}_2(\eta_1; \eta_2 \mid k) := \sum_{x_1 \in \eta_1} K^{(k)}(x_1; \eta_2), \quad 0 \le k \le n,$$

where $K^{(0)}(x_i; \eta_j)$, $x_i \in \eta_i$ and $i \neq j$ is given in (5.6) and $K^{(k)}(x_i; \eta_j)$, $x_i \in \eta_i$ and $i \neq j$ are defined as

$$K^{(k)}(x_i; \eta_j) := h^k \int_{\mathbb{R}^{dk}} \nu(x_i - y_1) \prod_{r=1}^{k-1} \nu(y_r - y_{r+1}) K^{(0)}(y_k; \eta_j)\, dy_1 \cdots dy_k, \quad k \ge 1. \tag{5.8}$$

Assume now that ν is polynomially bounded, i.e. $\nu(x) \le C\overline{\nu}(x)$ with $\overline{\nu}$ in (5.1) for some constant $C > 0$ and $\alpha > d$. Integrating over the points on the path from y_1 to y_{k-1}, Lemma 5.3 yields factors $2^{1+\alpha} C \overline{\nu}_1$

$$K^{(k)}(x_1; \eta_2) \le (hC)^k (2^{1+\alpha} \overline{\nu}_1)^{k-1} \int_{\mathbb{R}^d} \overline{\nu}(x_1 - y) K^{(0)}(y; \eta_2)\, dy, \quad k \ge 1. \tag{5.9}$$

Here, we also used the bound $\overline{\nu} \leq 1$. The integral in (5.9) can be estimated as follows. From (5.6),

$$K^{(0)}(y;\eta_i) \leq \sum_{x_i\in\eta_i} \sum_{\eta'\subset(\eta_i\setminus\{x_i\})} C^{|\eta'|+1}\overline{\nu}(y-x_i) \leq C(1+C)^{l_i-1} \sum_{x_i\in\eta_i} \overline{\nu}(y-x_i). \tag{5.10}$$

Inserting (5.10) in (5.9) and then using Lemma 5.3 again, we obtain the common upper bound

$$K^{(k)}(x_1;\eta_2) \leq (h\overline{\nu}_1 2^{1+\alpha}C)^k C(1+C)^{l_2-1} \sum_{x_2\in\eta_2} \overline{\nu}(x_1-x_2), \quad 0\leq k\leq n. \tag{5.11}$$

In summing over the trees connected to a single point of this path, the number of vertices in these trees is unlimited. This means that we can consider these trees individually, having base points on the $k+l$ points of the path from η_1 to η_2 and containing n_i+1 points ($i=1,\ldots,k+l$). There are $(n_i+1)^{n_i-1}$ such trees for each i, so we now have in total,

$$|\widetilde{\rho}_2^T(\eta_1;\eta_2)| \leq C(1+C)^{l_2-1}h^l \sum_{x_1\in\eta_1}\sum_{k=0}^{\infty}(h\overline{\nu}_1 2^{1+\alpha}C)^k \sum_{n_1,\ldots,n_{k+l}=0}^{\infty}\prod_{i=1}^{k+l}\frac{(n_i+1)^{n_i-1}(h\nu_1)^{n_i}}{n_i!}\sum_{x_2\in\eta_2}\overline{\nu}(x_1-x_2).$$

Here, there is a factor $\frac{n!}{k!n_1!\ldots n_{k+l}!}$ for the number of ways of distributing the vertices in γ over the individual trees and the remaining k points of γ and a factor $k!$ for the number of ways of ordering the vertices in the path connecting the two clusters as well as a factor $\frac{1}{n!}$ from the definition of the correlation function. Using now that $(k+1)^{k-1}\leq k!\mathrm{e}^k$ for $k\geq 1$, we obtain,

$$|\widetilde{\rho}_2^T(\eta_1;\eta_2)| \leq l_1l_2C(1+C)^{l_2-1}h^l\sum_{k=0}^{\infty}\left(h\overline{\nu}_1 2^{1+\alpha}C\right)^k\left(\sum_{n=0}^{\infty}(h\nu_1\mathrm{e})^n\right)^{k+l}\max_{x_1\in\eta_1;\,x_2\in\eta_2}\overline{\nu}(x_1-x_2).$$

Here, we assumed that $h(\nu_1\mathrm{e}+\overline{\nu}_1 2^{1+\alpha}C)<1$. Theorem 5.1 in the case $m=2$ is proven by setting

$$A_{2,1}=A_{2,2}:=\frac{1}{2}l_1l_2C(1+C)^{l_2-1}\left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^l\frac{1-h\nu_1\mathrm{e}}{1-h\nu_1\mathrm{e}-h\overline{\nu}_1 2^{1+\alpha}C}. \tag{5.12}$$

Remark 5.4 *Comparing the above formula with expression* (5.7)*, we have the remarkable identity,*

$$N_n^{(1)}(l)=\sum_{\substack{n_1,\ldots,n_l\geq 0\\ \sum_{i=1}^{l} n_i=n}}\frac{n!}{n_1!\cdots n_l!}\prod_{i=1}^{l}(n_i+1)^{n_i-1},$$

where we replaced $n-k$ by n and $k+l$ by l.

5.3 The case $m=3$.

Here, the situation is not too much more complicated. The cases where there is a line between at least one pair of η_1, η_2 and η_3 reduce to the case $m=2$. There remains the case that there is a tree on γ which is connected to all three. Again, this tree has only one point in γ which connects to η_i for each $i=1,2,3$, and by integrating out over intermediate y's which connect to only two others, this reduces to the case where these three points coincide. Assuming that the points connecting the tree to η_1, η_2 and η_3 are different points y_1, y_2 and y_3, there are 3 possible permutations of these points, and we can integrate out any intermediate points as before, yielding factors $2^{1+\alpha}\overline{\nu}_1$. In terms of Proposition 5.1, we have,

$$Q_3(\eta_1;\eta_2;\eta_3\mid n)=h^l\sum_{k=0}^{n}\binom{n}{k}k!N_{n-k}^{(1)}(l+k)(h\nu_1)^{n-k}\widetilde{Q}_3(\eta_1;\eta_2;\eta_3\mid k), \quad n\in\mathbb{N}_0, \tag{5.13}$$

where $\widetilde{Q}_3(\eta_1;\eta_2;\eta_3\mid k)$ is the contribution from all forest graphs in $\mathfrak{G}(\eta_1;\eta_2;\eta_3\mid\{y_1,\ldots,y_k\})$, in which all vertices y_i of γ are connected to at least two other vertices. Integrating out the vertices of γ connected to only 2 others yields factors $2^{1+\alpha}\overline{\nu}_1$ and results in a tree on γ where every vertex is connected to at least 3 others. There is only one such tree. It consists of a single point y of γ connected to η_1, η_2 and η_3. Conversely, given this tree, one can form trees with additional vertices connected to two points by adding a sequence of points between y and η_1, η_2 and η_3. In total, $\widetilde{Q}_3(\eta_1;\eta_2;\eta_3 \mid k)$ is the sum of 3 contributions

$$\widetilde{Q}_3(\eta_1;\eta_2;\eta_3 \mid k) = \widetilde{Q}_{3,1}(\eta_1;\eta_2;\eta_3 \mid k) + \widetilde{Q}_{3,2}(\eta_1;\eta_2;\eta_3 \mid k) + \widetilde{Q}_{3,3}(\eta_1;\eta_2;\eta_3 \mid k), \tag{5.14}$$

where $\widetilde{Q}_{3,3}(\eta_1;\eta_2;\eta_3 \mid k)$ contains the contributions of terms where there is no connection inside $\eta_1\cup\eta_2\cup\eta_3$ (3 components), $\widetilde{Q}_{3,2}(\eta_1;\eta_2;\eta_3 \mid k)$ corresponds to the terms where there is one or more line(s) between one pair of η_1, η_2 and η_3 (2 components), and $\widetilde{Q}_{3,1}(\eta_1;\eta_2;\eta_3 \mid k)$ contains the contributions where all 3 clusters are connected by lines inside $\eta_1\cup\eta_2\cup\eta_3$. In the latter, we must have $k=0$ since there cannot be another (outside) connection between two η_i's. From (5.14), (5.13) is the sum of three contributions

$$\begin{aligned}
Q_{3,1}(\eta_1;\eta_2;\eta_3 \mid n) :=& h^l \widetilde{Q}_{3,1}(\eta_1;\eta_2;\eta_3 \mid 0),\\
Q_{3,i}(\eta_1;\eta_2;\eta_3 \mid n) :=& h^l \sum_{k=0}^{n} \binom{n}{k} k! N^{(1)}_{n-k}(l+k)(h\nu_1)^{n-k}\widetilde{Q}_{3,i}(\eta_1;\eta_2;\eta_3 \mid k), \quad i=2,3.
\end{aligned}$$

In the case when all three clusters are connected,

$$\begin{aligned}
\widetilde{Q}_{3,1}(\eta_1;\eta_2;\eta_3 \mid 0) = \sum_{x_1\in\eta_1}\sideset{}{^*}\sum_{\eta_2'\subset\eta_2}\sideset{}{^*}\sum_{\eta_3'\subset\eta_3}\Bigg\{&\prod_{x_2\in\eta_2'}\nu(x_1-x_2)\sum_{x_1'\in\eta_1}\prod_{x_3\in\eta_3'}\nu(x_1'-x_3)\\
&+\prod_{x_2\in\eta_2'}\nu(x_1-x_2)\sum_{x_2'\in\eta_2}\prod_{x_3\in\eta_3'}\nu(x_2'-x_3)+\prod_{x_3\in\eta_3'}\nu(x_1-x_3)\sum_{x_3'\in\eta_3}\prod_{x_2\in\eta_2'}\nu(x_3'-x_2)\Bigg\}.
\end{aligned}$$

Assume that $\nu(x)\le C\overline{\nu}(x)$ with $\overline{\nu}$ as in (5.1) for some constant $C>0$ and $\alpha>d$. From (5.10) together with the following upper bound on the sum over trees

$$h^l\sum_{n=0}^{\infty}\frac{(h\nu_1)^n}{n!}N^{(1)}_n(l)\le h^l\sum_{n_1,\dots,n_l=0}^{\infty}\prod_{i=1}^{l}\frac{(h\nu_1)^{n_i}}{n_i!}(n_i+1)^{n_i-1}\le h^l\prod_{i=1}^{l}\sum_{n_i=0}^{\infty}(h\nu_1\mathrm{e})^{n_i}=\left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^l, \tag{5.15}$$

and

$$\sum_{x_i\in\eta_i}\sum_{x_j\in\eta_j}\overline{\nu}(x_i-x_j)\le l_il_j\max_{x_i\in\eta_i;\,x_j\in\eta_j}\overline{\nu}(x_i-x_j),\quad i\ne j,$$

we then obtain,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{n!}Q_{3,1}(\eta_1;\eta_2,\eta_3 \mid n)\\
&\le l_1l_2l_3C^2(1+C)^{l_2+l_3-2}\left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^l\Bigg\{l_1\max_{x_1\in\eta_1;\,x_2\in\eta_2}\overline{\nu}(x_1-x_2)\max_{x_1\in\eta_1;\,x_3\in\eta_3}\overline{\nu}(x_1-x_3)\\
&\quad+l_2\max_{x_1\in\eta_1;\,x_2\in\eta_2}\overline{\nu}(x_1-x_2)\max_{x_2\in\eta_2;\,x_3\in\eta_3}\overline{\nu}(x_2-x_3)+l_3\max_{x_1\in\eta_1;\,x_3\in\eta_3}\overline{\nu}(x_1-x_3)\max_{x_3\in\eta_3;\,x_2\in\eta_2}\overline{\nu}(x_3-x_2)\Bigg\}\\
&\le l_1l_2l_3(l_1+l_2+l_3)C^2(1+C)^{l_2+l_3-2}\left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^l\max_{T_3\in\mathcal{T}_3}\overline{\nu}_{T_3}.
\end{aligned}\tag{5.16}$$

In the cases where only one pair of η_1, η_2 and η_3 are connected, we have, for all $0\le k\le n$,

$$\begin{aligned}
\widetilde{Q}_{3,2}(\eta_1;\eta_2;\eta_3 \mid k) := \sum_{x_1\in\eta_1}\Bigg\{&K^{(0)}(x_1;\eta_2)\sum_{x\in\eta_1\cup\eta_2}K^{(k)}(x;\eta_3)+K^{(0)}(x_1;\eta_3)\sum_{x\in\eta_1\cup\eta_3}K^{(k)}(x;\eta_2)\\
&+K^{(k)}(x_1;\eta_2)\sum_{x_2\in\eta_2}K^{(0)}(x_2,\eta_3)+K^{(k)}(x_1;\eta_3)\sum_{x_3\in\eta_3}K^{(0)}(x_3;\eta_2)\Bigg\},
\end{aligned}$$

where the kernels $K^{(0)}$ and $K^{(k)}$, $k \geq 1$ are defined in (5.6) and (5.8) respectively. As in the case $m = 2$, we now use the assumption that $\nu(x) \leq C\overline{\nu}(x)$ with $\overline{\nu}$ as in (5.1) for $\alpha > d$ and some constant $C > 0$. Then the kernels $K^{(k)}$, $0 \leq k \leq n$ are estimated as in (5.11). Therefore,

$$\begin{aligned}
&\widetilde{Q}_{3,2}(\eta_1;\eta_2;\eta_3 \mid k) \leq l_1 C^2(1+C)^{l_2+l_3-2}(h\overline{\nu}_1 2^{1+\alpha}C)^k \\
&\times \Big\{ l_2(l_1+l_2)l_3 \max_{x_1\in\eta_1;\, x_2\in\eta_2} \overline{\nu}(x_1-x_2) \max_{x\in\eta_1\cup\eta_2;\, x_3\in\eta_3} \overline{\nu}(x-x_3) \\
&\quad + l_3(l_1+l_3)l_2 \max_{x_1\in\eta_1;\, x_3\in\eta_3} \overline{\nu}(x_1-x_3) \max_{x\in\eta_1\cup\eta_3;\, x_2\in\eta_2} \overline{\nu}(x-x_2) \\
&\quad + l_2^2 l_3 \max_{x_1\in\eta_1;\, x_2\in\eta_2} \overline{\nu}(x_1-x_2) \max_{x_2'\in\eta_2;\, x_3\in\eta_3} \overline{\nu}(x_2'-x_3) + l_3^2 l_2 \max_{x_1\in\eta_1;\, x_3\in\eta_3} \overline{\nu}(x_1-x_3) \max_{x_3'\in\eta_3;\, x_2\in\eta_2} \overline{\nu}(x_2-x_3') \Big\}.
\end{aligned}$$

It remains to sum over the external trees, see (5.15), and we obtain,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n!} Q_{3,2}(\eta_1;\eta_2;\eta_3 \mid n) \\
&\qquad \leq 2l_1l_2l_3(l_1+l_2+l_3)C^2(1+C)^{l_2+l_3-2}\left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^l \frac{h\overline{\nu}_1 2^{1+\alpha}C}{1-h\nu_1\mathrm{e}-h\overline{\nu}_1 2^{1+\alpha}C} \max_{T_3\in\mathcal{T}_3} \overline{\nu}_{T_3}. \qquad (5.17)
\end{aligned}$$

Here, we assumed that $h(\nu_1\mathrm{e} + \overline{\nu}_1 2^{1+\alpha}C) < 1$.

There remains the case where there is no line between any points of $\eta_1 \cup \eta_2 \cup \eta_3$. As explained earlier,

$$\widetilde{Q}_{3,3}(\eta_1;\eta_2;\eta_3 \mid k) = \sum_{x_1\in\eta_1} \sum_{\substack{k_1,k_2,k_3\geq 0\\ k_1+k_2+k_3=k-1}} h\int_{\mathbb{R}^d} K^{(k_1)}(x_1,y)K^{(k_2)}(y;\eta_2)K^{(k_3)}(y;\eta_3)\,dy, \quad 0\leq k\leq n,$$

where we set $K^{(k)}(x,y) := K^{(k)}(y;\{x\})$. Inserting the bound (5.11), we get,

$$\begin{aligned}
&\widetilde{Q}_{3,3}(\eta_1;\eta_2;\eta_3 \mid k) \\
&\quad \leq C^3(1+C)^{l_2+l_3-2} \sum_{x_1\in\eta_1} \sum_{\substack{k_1,k_2,k_3\geq 0\\ k_1+k_2+k_3=k-1}} (h\overline{\nu}_1 2^{1+\alpha}C)^{k-1} h\int_{\mathbb{R}^d} \overline{\nu}(x_1-y)\prod_{i=2}^{3}\sum_{x_i\in\eta_i} \overline{\nu}(x_i-y)\,dy \\
&\quad \leq l_1l_2l_3 2^{2\alpha}C^3(1+C)^{l_2+l_3-2}h\overline{\nu}_1 \sum_{\substack{k_1,k_2,k_3\geq 0\\ k_1+k_2+k_3=k-1}} (h\overline{\nu}_1 2^{1+\alpha}C)^{k-1} \\
&\qquad \times \max_{x_1\in\eta_1;\, x_2\in\eta_2;\, x_3\in\eta_3} \{\overline{\nu}(x_1-x_2)\overline{\nu}(x_1-x_3) + \overline{\nu}(x_1-x_2)\overline{\nu}(x_2-x_3) + \overline{\nu}(x_1-x_3)\overline{\nu}(x_2-x_3)\}.
\end{aligned}$$

Summing over trees attached to points of these paths, summing over $n' := n-k$ and using (5.15),

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n!} Q_{3,3}(\eta_1;\eta_2;\eta_3 \mid n) \\
&\leq 3l_1l_2l_3 2^{2\alpha}C^3(1+C)^{l_2+l_3-2} \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{h^l(h\overline{\nu}_1)}{(n-k)!} N^{(1)}_{n-k}(k+l)(h\nu_1)^{n-k} \sum_{\substack{k_1,k_2,k_3\geq 0\\ \sum_{i=1}^{3} k_i=k-1}} (h\overline{\nu}_1 2^{1+\alpha}C)^{k-1} \max_{T_3\in\mathcal{T}_3} \overline{\nu}_{T_3} \\
&\leq 3l_1l_2l_3 2^{2\alpha}C^3(1+C)^{l_2+l_3-2} \frac{h^l(h\overline{\nu}_1)}{(1-h\nu_1\mathrm{e})^{l+1}} \sum_{k=1}^{\infty} \sum_{\substack{k_1,k_2,k_3\geq 0\\ \sum_{i=1}^{3} k_i=k-1}} \left(\frac{h\overline{\nu}_1 2^{1+\alpha}C}{1-h\nu_1\mathrm{e}}\right)^{k-1} \max_{T_3\in\mathcal{T}_3} \overline{\nu}_{T_3} \\
&\leq 3l_1l_2l_3 2^{2\alpha}C^3(1+C)^{l_2+l_3-2} \left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^l \frac{h\overline{\nu}_1(1-h\nu_1\mathrm{e})^2}{(1-h\nu_1\mathrm{e}-h\overline{\nu}_1 2^{1+\alpha}C)^3} \max_{T_3\in\mathcal{T}_3} \overline{\nu}_{T_3}. \qquad (5.18)
\end{aligned}$$

In view of (5.16), (5.17) and (5.18), Theorem 5.1 in the case $m = 3$ is proven by setting

$$A_{3,1} := l_1 l_2 l_3 l C^2 (1+C)^{l_2+l_3-2} \left(\frac{h}{1-h\nu_1 \mathrm{e}} \right)^l , \tag{5.19}$$

$$A_{3,2} := 2 l_1 l_2 l_3 l C^2 (1+C)^{l_2+l_3-2} \left(\frac{h}{1-h\nu_1 \mathrm{e}} \right)^l \frac{h\overline{\nu}_1 2^{1+\alpha} C}{1-h\nu_1 \mathrm{e} - h\overline{\nu}_1 2^{1+\alpha} C}, \tag{5.20}$$

$$A_{3,3} := 3 l_1 l_2 l_3 2^{2\alpha} C^3 (1+C)^{l_2+l_3-2} \left(\frac{h}{1-h\nu_1 \mathrm{e}} \right)^l \frac{h\overline{\nu}_1 (1-h\nu_1 \mathrm{e})^2}{(1-h\nu_1 \mathrm{e} - h\overline{\nu}_1 2^{1+\alpha} C)^3}. \tag{5.21}$$

5.4 The case of general m.

As before, we integrate out intermediate points y, which connect to only two others (as well as trees of points y connected to a single point of $\bigcup_{i=1}^m \eta_i$). We are then left with trees where each y has order ≥ 3. In terms of Proposition 5.1, we have,

$$Q_m(\eta_1; \dots; \eta_m \mid n) = h^l \sum_{k=0}^n \binom{n}{k} k! N^{(1)}_{n-k}(l+k) (h\nu_1)^{n-k} \widetilde{Q}_m(\eta_1; \dots; \eta_m \mid k), \quad n \in \mathbb{N}_0, \tag{5.22}$$

with $\widetilde{Q}_m(\eta_1; \dots; \eta_m \mid k)$ the contributions from trees with k vertices in γ, each of which has order ≥ 2.

Denote by σ the number of connected components in $\bigcup_{i=1}^m \eta_i$. If $\sigma = 1$, i.e. all the η_i are connected directly, then there is no such tree, and the contribution is only from $k = 0$

$$\widetilde{Q}_{m,1}(\eta_1; \dots; \eta_m \mid 0) = \sum_{\widetilde{f} \in \mathfrak{S}(\eta_1; \dots; \eta_m \mid \emptyset)} \prod_{(i,j) \in \widetilde{f}} \nu(x_i - x_j).$$

This can be bounded by $C^{l-l_1} N_0^{(m)}(l_1; \dots; l_m) \times \max_{T_m \in \mathcal{T}_m} \overline{\nu}_{T_m}$, but a better bound is obtained as in the case $m = 3$, replacing the factor L_i in $N_0^{(m)}(l_1; \dots; l_m)$ by the sum in (5.10)

$$\widetilde{Q}_{m,1}(\eta_1; \dots; \eta_m \mid 0) \leq l^{m-2} (\prod_{i=1}^m l_i) C^{m-1} (1+C)^{l-l_1-m+1} \max_{T_m \in \mathcal{T}_m} \overline{\nu}_{T_m}.$$

By using the estimate (5.15), we obtain,

$$\sum_{n=0}^\infty \frac{1}{n!} Q_{m,1}(\eta_1; \dots, \eta_m \mid n) \leq A_{m,1} \max_{T_m \in \mathcal{T}_m} \overline{\nu}_{T_m},$$

$$A_{m,1} := l^{m-2} (\prod_{i=1}^m l_i) C^{m-1} (1+C)^{l-l_1-m+1} \left(\frac{h}{1-h\nu_1 \mathrm{e}} \right)^l , \tag{5.23}$$

which agrees with (5.16) in case $m = 3$. If $\sigma = 2$, the only possible such tree is a chain connecting one component to the other. In this case, we have $k \geq 1$ and,

$$\begin{aligned}
&\widetilde{Q}_{m,2}(\eta_1; \dots; \eta_m \mid k) \\
&= \sum_{\substack{I_1 \subset \{1,\dots,m\} \\ 1 \in I_1,\, |I_1| < m}} \sum_{x_1 \in \eta_{I_1}} \sum_{\substack{i_2 \in I_2 \\ I_2 = I_1^c}} K^{(k)}(x_1; \eta_{i_2}) \\
&\qquad \times \sum_{f_1 \in \mathfrak{S}(\eta_1; \eta_{I_1 \setminus \{1\}} \mid \emptyset)} \prod_{(x,x') \in f_1} \nu(x-x') \sum_{f_2 \in \mathfrak{S}(\eta_{i_2}; \eta_{I_2 \setminus \{i_2\}} \mid \emptyset)} \prod_{(x,x') \in f_2} \nu(x-x') \\
&\leq C (h\overline{\nu}_1 2^{1+\alpha} C)^k \sum_{\substack{\{I_1, I_2\} \in \Pi_2(\{1,\dots,m\}) \\ 1 \in I_1}} \sum_{i_2 \in I_2} (1+C)^{l_{i_2}-1} \sum_{x_1 \in \eta_{I_1}} \sum_{x_2 \in \eta_{i_2}} \max_{x_2 \in \eta_{i_2}} \overline{\nu}(x_1 - x_2) \\
&\qquad \times \sum_{f_1 \in \mathfrak{S}(\eta_1; \eta_{I_1 \setminus \{1\}} \mid \emptyset)} \prod_{(x,x') \in f_1} \nu(x-x') \sum_{f_2 \in \mathfrak{S}(\eta_{i_2}; \eta_{I_2 \setminus \{i_2\}} \mid \emptyset)} \prod_{(x,x') \in f_2} \nu(x-x').
\end{aligned}$$

Here, $\Pi_2(\{1,\dots,m\})$ denotes the set of all partitions of $\{1,\dots,m\}$ into 2 non-empty subsets. The latter sums over forests can be estimated, replacing again L_i by the sum in (5.10), as

$$\sum_{f_j\in\mathfrak{S}(\eta_{i_j};\eta_{I_j\setminus\{i_j\}}|\emptyset)}\prod_{(x,x')\in f_j}\nu(x-x')\le l_{I_j}^{|I_j|-2}(\prod_{i\in I_j}l_i)C^{|I_j|-1}(1+C)^{\sum_{i\in I_j\setminus\{i_j\}}(l_i-1)}\max_{T_j\in\mathcal{T}(I_j)}\overline{\nu}_{T_j}. \tag{5.24}$$

Together with the link (x_1,x_2), we obtain a tree on $\{1,\dots,m\}$ (below we denote $l_I:=\sum_{i\in I}l_i$)

$$\widetilde{Q}_{m,2}(\eta_1;\dots;\eta_m\mid k)\le(\prod_{i=1}^m l_i)\sum_{\substack{\{I_1,I_2\}\in\Pi_2(\{1,\dots,m\})\\1\in I_1}}\prod_{i=1}^2 l_{I_i}^{|I_i|-1}C^{m-1}(1+C)^{l-l_1-m+1}(h\overline{\nu}_1 2^{1+\alpha}C)^k\max_{T_m\in\mathcal{T}_m}\overline{\nu}_{T_m}.$$

The sum over partitions can in fact be evaluated (see [9]) and yields,

$$\sum_{\substack{\{I_1,I_2\}\in\Pi_2(\{1,\dots,m\})\\1\in I_1}}\prod_{i=1}^2 l_{I_i}^{|I_i|-1}=(m-1)l^{m-2},\quad m\ge 2.$$

It remains to sum over n and k. Hence,

$$\sum_{n=1}^\infty\frac{1}{n!}Q_{m,2}(\eta_1;\dots;\eta_m\mid n)\le A_{m,2}\max_{T_m\in\mathcal{T}_m}\overline{\nu}_{T_m},$$

$$A_{m,2}:=(m-1)(\prod_{i=1}^m l_i)l^{m-2}C^{m-1}(1+C)^{l-l_1-m+1}\left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^l\frac{h\overline{\nu}_1 2^{1+\alpha}C}{1-h\nu_1\mathrm{e}-h\overline{\nu}_1 2^{1+\alpha}C}. \tag{5.25}$$

Here, we assumed that $h(\nu_1\mathrm{e}+\overline{\nu}_1 2^{1+\alpha}C)<1$. Note that this agrees with (5.17) in case $m=3$.

For $\sigma\ge 3$, there is at least one point $y\in\gamma$ which is connected to more than two other vertices. The tree on γ connecting the different components can again be reduced to a tree T where all vertices have order ≥ 3 by integrating out the vertices y of order 2, yielding factors $K^{(k)}$. The number of such vertices in γ is at most $\sigma-2$, where σ is the number of connected components of $\bigcup_{i=1}^m\eta_i$. The reduction formula reads

$$\widetilde{Q}_{m,\sigma}(\eta_1;\dots;\eta_m\mid k)=\sum_{r=1}^{\min\{k,\sigma-2\}}\sum_{T\in\mathcal{T}_r}\sum_{\substack{\{I_j\}_{j=1}^\sigma\in\Pi_\sigma(\{1,\dots,m\})\\1\in I_1}}\sum_{\pi\in\mathcal{M}^{(3)}(T,\sigma,r)}\widetilde{Q}_{\{I_j\}_{j=1}^\sigma,T,\pi}(r),\quad k\ge 1, \tag{5.26}$$

with

$$\begin{aligned}\widetilde{Q}_{\{I_j\}_{j=1}^\sigma,T,\pi}(r):=&\frac{h^r}{k!}\binom{k}{r}(k-r)!\int_{\mathbb{R}^{dr}}dy_1\cdots dy_r\sum_{\substack{(k_{y,y'})_{(y,y')\in T}\\k_{y,y'}\ge 0,\ \sum_{(y,y')\in T}k_{y,y'}\le k-r}}\prod_{(y,y')\in T}K^{(k_{y,y'})}(y,y')\\&\times\sum_{\substack{k_1,\dots,k_\sigma\ge 0\\\sum_{j=1}^\sigma k_j+\sum_{(y,y')\in T}k_{y,y'}=k-r}}\sum_{x_1\in\eta_1}K^{(k_1)}(x_1,y_{\pi(1)})\times\sum_{f_1\in\mathfrak{S}(\eta_1;\eta_{I_1\setminus\{1\}}|\emptyset)}\prod_{(x,x')\in f_1}\nu(x-x')\\&\times\prod_{j=2}^\sigma\left(\sum_{i_j\in I_j}K^{(k_j)}(y_{\pi(j)};\eta_{i_j})\sum_{f_j\in\mathfrak{S}(\eta_{i_j};\eta_{I_j\setminus\{i_j\}}|\emptyset)}\prod_{(x,x')\in f_j}\nu(x-x')\right).\end{aligned}$$

In (5.26), $\mathcal{T}_r$ denotes the set of tree graphs on r points, I_j the set of i such that η_i belongs to the j-th component, $\Pi_\sigma(\{1,\dots,m\})$ the set of all partitions $\{I_j\}_{j=1}^\sigma$ of $\{1,\dots,m\}$ into σ non-empty subsets and $\mathcal{M}^{(3)}(T,\sigma,r)$ the set of maps $\pi:\{1,\dots,\sigma\}\to\{1,\dots,r\}$ such that $|\{y\in T:(y_i,y)\in T\}|+|\pi^{-1}(i)|\ge 3$, $i=1,\dots,r$, i.e., each point y_i has at least 3 lines attached in the resulting graph. $\pi\in\mathcal{M}^{(3)}(T,\sigma,r)$ determines the points of attachment of each component to the tree T. We point out that the factor $1/k!$ compensates for $k!$ in (5.22), and the factors $\binom{k}{r}$ and $(k-r)!$ then count the number of ways of choosing which y_i are associated with the vertices of T and the number of ways of distributing the remaining y_i over the vertices of order 2. If q is the number of vertices $y \in \gamma$ of T connected to at least 3 other points of γ, then the tree T determines $q-1$ lines between these vertices. In addition, there are $q_e \geq 3q - 2(q-1) = q+2$ endpoints. Each intermediate point of the tree must be connected to at least one components of $\bigcup_{i=1}^m \eta_i$, whereas each endpoint must be connected to at least two. Let t be the number of intermediate points. Then $\sigma \geq t + 2q_e$ and, as result, $r = q_e + q + t \leq 2q_e - 2 + t \leq \sigma - 2$. It follows that (5.26) can be rewritten as

$$\widetilde{Q}_{m,\sigma}(\eta_1;\ldots;\eta_m \mid k) = \sum_{r=1}^{\min\{k,\sigma-2\}} \sum_{T\in\mathcal{T}_r} \sum_{\substack{\{I_j\}_{j=1}^{\sigma}\in\Pi_\sigma(\{1,\ldots,m\})\\ 1\in I_1}} \sum_{\pi\in\mathcal{M}^{(3)}(T,\sigma,r)} \widetilde{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r). \tag{5.27}$$

The contribution of a given tree $T \in \mathcal{T}_r$ with r vertices and an assignment π is bounded above by

$$\begin{aligned}\widetilde{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r) \leq \frac{h^r}{r!} C^{\sigma+r-1}(h\overline{\nu}_1 2^{1+\alpha} C)^{k-r} & \sum_{\substack{(k_{y,y'})_{(y,y')\in T}\\ k_{y,y'}\geq 0,\ \sum_{(y,y')\in T} k_{y,y'}\leq k-r}} \sum_{\substack{k_1,\ldots,k_\sigma\geq 0\\ \sum_{j=1}^{\sigma} k_j + \sum_{(y,y')\in T} k_{y,y'}=k-r}} \\ \times \int_{\mathbb{R}^{dr}} dy_1\cdots dy_r \prod_{(y,y')\in T} \overline{\nu}(y-y') & \sum_{x_1\in\eta_{I_1}} \overline{\nu}(x_1 - y_{\pi(1)}) \sum_{f_1\in\mathfrak{S}(\eta_1;\eta_{I_1\setminus\{1\}}|\emptyset)} \prod_{(x,x')\in f_1} \nu(x-x') \\ \times \prod_{j=2}^{\sigma} \left(\sum_{i_j\in I_j} (1+C)^{l_{i_j}-1} \sum_{x_j\in\eta_{i_j}} \overline{\nu}(x_j - y_{\pi(j)}) \right. & \left. \sum_{f_j\in\mathfrak{S}(\eta_{i_j};\eta_{I_j\setminus\{i_j\}}|\emptyset)} \prod_{(x,x')\in f_j} \nu(x-x') \right).\end{aligned}$$

Inserting the bound (5.24) which holds for all $j \in \{1,\ldots,\sigma\}$, we have,

$$\begin{aligned}\widetilde{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r) \leq \frac{h^r}{r!} C^{\sigma+r-1}(h\overline{\nu}_1 2^{1+\alpha} C)^{k-r} & \sum_{\substack{(k_{y,y'})_{(y,y')\in T}\\ k_{y,y'}\geq 0,\ \sum_{(y,y')\in T} k_{y,y'}\leq k-r}} \sum_{\substack{k_1,\ldots,k_\sigma\geq 0\\ \sum_{j=1}^{\sigma} k_j + \sum_{(y,y')\in T} k_{y,y'}=k-r}} \\ \times \int_{\mathbb{R}^{dr}} dy_1\cdots dy_r \prod_{(y,y')\in T} \overline{\nu}(y-y') & \left(\sum_{x_1\in\eta_{I_1}} \overline{\nu}(x_1 - y_{\pi(1)}) l_{I_1}^{|I_1|-2} C^{|I_1|-1} (1+C)^{\sum_{i\in I_1\setminus\{1\}}(l_i-1)} \max_{T_1\in\mathcal{T}(I_1)} \overline{\nu}_{T_1} \right) \\ \times \prod_{j=2}^{\sigma} & \left(\sum_{x_j\in\eta_{I_j}} \overline{\nu}(x_j - y_{\pi(j)}) l_{I_j}^{|I_j|-2} C^{|I_j|-1} (1+C)^{\sum_{i\in I_j}(l_i-1)} \max_{T_j\in\mathcal{T}(I_j)} \overline{\nu}_{T_j} \right).\end{aligned} \tag{5.28}$$

Subsequently, we need to bound the integrals

$$\int_{\mathbb{R}^{dr}} \prod_{(y,y')\in T} \overline{\nu}(y-y') \prod_{j=1}^{\sigma} \overline{\nu}(x_j - y_{\pi(j)})\, dy_1\cdots dy_r,$$

where $x_j \in \eta_{I_j}$ and (y,y') is a line in T between y_i and y_j for some $i,j = 1,\ldots,r$. Note that it is allowed for $y_{\pi(j)}$ to be equal to $y_{\pi(j')}$ with $j \neq j'$. However, the number of unequal $y_{\pi(j)}$ must be at least twice the number of endpoints of the graph T on γ. To estimate this integral, we integrate over the endpoints of T except the endpoint $\pi(1)$ connected to I_1. Integrating over an endpoint y, Lemma 5.3 yields

$$\int_{\mathbb{R}^d} \prod_{i=1}^{p} \overline{\nu}(x_i - y)\overline{\nu}(y-y')\, dy \leq 2^{\alpha p}\overline{\nu}_1 \left(\sum_{i=1}^{p} \prod_{\substack{k=1\\ k\neq i}}^{p} \overline{\nu}(x_i - x_k)\overline{\nu}(x_i - y') + \prod_{i=1}^{p} \overline{\nu}(x_i - y') \right).$$

Therefore, setting $p = |\pi^{-1}(i)|$, we have,

$$\int_{\mathbb{R}^d} \overline{\nu}(y_i - y_i') \prod_{j\in\pi^{-1}(i)} \overline{\nu}(x_j - y_i)\, dy_i$$
$$\leq 2^{\alpha|\pi^{-1}(i)|}\overline{\nu}_1 \left(\sum_{j\in\pi^{-1}(i)} \prod_{\substack{j'\in\pi^{-1}(i)\\ j'\neq j}} \overline{\nu}(x_j - x_{j'})\overline{\nu}(x_j - y_i') + \prod_{j\in\pi^{-1}(i)} \overline{\nu}(x_j - y_i') \right).$$

Inserting this into (5.28), the first term in brackets combines trees T_j on clusters connected to the same endpoint y_i, i.e. $\pi(j) = i$, into a single tree T_i' connected to y_i'. The second term connects all x_j with $\pi(j) = i$ to y_i'. Note also that the factors $C^{|I_j|-1}$ combine to give $\prod_{j=1}^{\sigma} C^{|I_j|-1} = C^{m-\sigma}$ and similarly,

$$\prod_{j=1}^{\sigma} (1+C)^{\sum\limits_{i\in I_j\setminus\{1\}}(l_i-1)} = (1+C)^{l-l_1-m+1}.$$

Thus, we obtain,

$$\widetilde{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r) \leq \frac{h^r}{r!}\Big(\prod_{j=1}^{\sigma} l_{I_j}^{|I_j|-2}\Big)C^{m+r-1}(1+C)^{l-l_1-m+1}(h\overline{\nu}_1 2^{1+\alpha}C)^{k-r}$$
$$\times \Big|\{(k_i)_{i=1}^{\sigma+r-1} : \sum_{i=1}^{\sigma+r-1} k_i = k-r\}\Big|\, \overline{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r), \quad (5.29)$$

where it is understood that $k_i \geq 0$ for $i = 1, \ldots, \sigma + r - 1$, and where

$$\overline{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r) := \int_{\mathbb{R}^{dr}} \prod_{(y,y')\in T} \overline{\nu}(y - y') \left(\prod_{j=1}^{\sigma} \sum_{x_j\in\eta_{I_j}} \overline{\nu}(x_j - y_{\pi(j)}) \max_{T_j\in\mathcal{T}(I_j)} \overline{\nu}_{T_j} \right) dy_1 \cdots dy_r. \quad (5.30)$$

Singling out the endpoints ∂T of T other than $\pi(1)$ and denoting $\Lambda(\pi, T) := \pi^{-1}(\partial T \setminus \{\pi(1)\})$, (5.30) can be rewritten as

$$\overline{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r) = \int_{\mathbb{R}^{dr}} dy_1 \cdots dy_r \prod_{(y,y')\in((T\setminus\partial T)\cup\{\pi(1)\})} \overline{\nu}(y - y') \sum_{x_1\in\eta_{I_1}} \overline{\nu}(x_1 - y_{\pi(1)}) \max_{T_1\in\mathcal{T}(I_1)} \overline{\nu}_{T_1}$$
$$\times \prod_{\substack{j>1\\ \pi(j)\in(T\setminus\partial T)}} \left(\sum_{x_j\in\eta_{I_j}} \overline{\nu}(x_j - y_{\pi(j)}) \max_{T_j\in\mathcal{T}(I_j)} \overline{\nu}_{T_j} \right)$$
$$\times \sum_{\substack{\{x_j\}_{j\in\Lambda(\pi,T)}\\ x_j\in\eta_{I_j}}} \prod_{j\in\Lambda(\pi,T)} \overline{\nu}(x_j - y_{\pi(j)})\overline{\nu}(y_{\pi(j)} - y'_{\pi(j)}) \max_{T_j\in\mathcal{T}(I_j)} \overline{\nu}_{T_j}.$$

Integrating out the endpoints $y_{\pi(j)}$, we have,

$$\overline{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r) \leq \int_{\mathbb{R}^{d|T\setminus\partial T|}} \prod_{i\in(T\setminus\partial T)} dy_i \prod_{(y,y')\in((T\setminus\partial T)\cup\{\pi(1)\})} \overline{\nu}(y - y') \sum_{x_1\in\eta_{I_1}} \overline{\nu}(x_1 - y_{\pi(1)}) \max_{T_1\in\mathcal{T}(I_1)} \overline{\nu}_{T_1}$$
$$\times \prod_{\substack{j>1\\ \pi(j)\in(T\setminus\partial T)}} \left(\sum_{x_j\in\eta_{I_j}} \overline{\nu}(x_j - y_{\pi(j)}) \max_{T_j\in\mathcal{T}(I_j)} \overline{\nu}_{T_j} \right) \prod_{i\in(\partial T\setminus\{\pi(1)\})} \Bigg\{ 2^{\alpha|\pi^{-1}(i)|}\overline{\nu}_1 \sum_{\substack{\{x_j\}_{j\in\pi^{-1}(i)}\\ x_j\in\eta_{I_j}}}$$
$$\times \left(\sum_{j\in\pi^{-1}(i)} \prod_{\substack{j'\in\pi^{-1}(i)\\ j'\neq j}} \overline{\nu}(x_j - x_{j'})\overline{\nu}(x_j - y_i') + \prod_{j\in\pi^{-1}(i)} \overline{\nu}(x_j - y_i') \right) \max_{T_j\in\mathcal{T}(I_j)} \overline{\nu}_{T_j} \Bigg\}. \quad (5.31)$$

After this first integration, some of the neighbours y_i' have become endpoints of a reduced tree T. We integrate out these points next and proceed this way until T is reduced to a single point. We can write the expression (5.31) in terms of the reduced tree as follows

$$\overline{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r) \le 2^{\alpha|\pi^{-1}(\partial T\setminus\{\pi(1)\})|}\overline{\nu}_1^{|\partial T\setminus\{\pi(1)\}|} \sum_{\substack{\{I_j'\}_{j=1}^{\sigma'}\in\Pi_{\sigma'}(\{1,\dots,m\})\\ 1\in I_1'}} \sum_{\pi'} \overline{Q}_{\{I_j'\}_{j=1}^{\sigma'},T\setminus\partial T,\pi'}(r), \tag{5.32}$$

where for each $i' \in ((T\setminus\partial T)\cup\{\pi(1)\})$, the set of j' such that $\pi'(j') = i'$ is given by

$$(\pi')^{-1}(i') = \pi^{-1}(i') \cup \bigcup_{\substack{i\in\partial T\\ y_i'=y_{i'}}} S_i,$$

where, either $S_i := \{j\}$ for some $j \in \pi^{-1}(i)$, or $S_i := \pi^{-1}(i)$. These two cases respectively correspond to the two terms in the last factor of the right-hand side of (5.31). In the first case, the trees T_j with $\pi(j) = i$ combine into a single tree

$$T_j' := \bigcup_{j'\in\pi^{-1}(i)} T_{j'} \cup \{(x_j, x_{j'}) : j' \in \pi^{-1}(i),\, j' \ne j\}.$$

In the second case, the forests f_j are unchanged. The number of components is reduced to

$$\sigma' = \sigma - \sum_{\substack{i\in\partial T\\ |S_i|=1}} (|\pi^{-1}(i)| - 1).$$

The corresponding subdivision is $I_j' = \bigcup_{j'\in\pi^{-1}(i)} I_{j'}$ in the first case, and $I_j' = I_j$ for all $j \in \pi^{-1}(i)$ in the second case. In any case, we obviously have $I_j' = I_j$ for $j \in \pi^{-1}(i')$. We stress the point that the definition of $\overline{Q}$ in the right-hand side of (5.32) has been slightly modified: we replaced $\overline{\nu}_{T_j}$ by the quantity

$$\widetilde{\nu}_{T_j'} := \prod_{\substack{j'\in\pi^{-1}(i)\\ j'\ne j}} l_{I_{j'}}\overline{\nu}_{T_j'} \quad \text{if } S_i = \{j\}.$$

After at most $r-1$ stages, the forest graph reduces to a single point $r = 1$. At the final stage, we need to integrate over the last vertex $y = y_{\pi(1)}$. Noticing that $I_1' = I_1$, it follows,

$$\begin{aligned}\overline{Q}_{\{I_j'\}_{j=1}^{\sigma'},\{y\},1}(r) &= \int_{\mathbb{R}^d} \sum_{x_1\in I_1'} \overline{\nu}(x_1-y) \max_{T_1\in\mathcal{T}(I_1)} \overline{\nu}_{T_1} \sum_{\substack{\{x_j\}_{j=2}^{\sigma'}\\ x_j\in\eta_{I_j'}}} \prod_{j=2}^{\sigma'} \overline{\nu}(x_j-y) \max_{T_j'\in\mathcal{T}(I_j')} \widetilde{\nu}_{T_j'} \\ &\le 2^{\alpha(\sigma'-1)}\overline{\nu}_1 \sum_{\substack{\{x_j\}_{j=1}^{\sigma'}\\ x_j\in\eta_{I_j'}}} \sum_{j=1}^{\sigma'} \left(\prod_{\substack{j'=1\\ j'\ne j}}^{\sigma'} \overline{\nu}(x_j-x_{j'}) \max_{T\in\mathcal{T}(\{1,\dots,m\}\setminus\{j\})} \widetilde{\nu}_T \right) \le 2^{\alpha(\sigma'-1)}\overline{\nu}_1 (\prod_{j=1}^{\sigma} l_{I_j}) \max_{T_m\in\mathcal{T}_m} \overline{\nu}_{T_m}.\end{aligned}$$

Since all trees generated are distinct, we can write, bounding σ' by σ at each stage,

$$\overline{Q}_{\{I_j\}_{j=1}^{\sigma},T,\pi}(r) \le 2^{\alpha(\sigma r-1)}\overline{\nu}_1^r (\prod_{j=1}^{\sigma} l_{I_j}) \max_{T_m\in\mathcal{T}_m} \overline{\nu}_{T_m}.$$

Inserting this in (5.29) and (5.27), we obtain the upper bound

$$\begin{aligned}\widetilde{Q}_{m,\sigma}(\eta_1;\dots;\eta_m \mid k) \le &\sum_{\substack{\{I_j\}_{j=1}^{\sigma}\in\Pi_\sigma(\{1,\dots,m\})\\ 1\in I_1}} (\prod_{j=1}^{\sigma} l_{I_j}^{|I_j|-1}) C^{m-1}(1+C)^{l-l_1-m+1}(h\overline{\nu}_1 C)^k \\ &\times \sum_{r=1}^{\min\{k,\sigma-2\}} \sum_{T\in\mathcal{T}_r} \sum_{\pi\in\mathcal{M}^{(3)}(T,\sigma,r)} \frac{1}{r!} 2^{\alpha(r\sigma-1)} 2^{(1+\alpha)(k-r)} \big|\{(k_i)_{i=1}^{\sigma+r-1} : \sum_{i=1}^{\sigma+r-1} k_i = k-r\}\big| \max_{T_m\in\mathcal{T}_m} \overline{\nu}_{T_m},\end{aligned}$$

where it is understood that $k_i \geq 0$ for $i = 1, \ldots, \sigma + r - 1$. We now bound the number of trees $T \in \mathcal{T}_r$ by $r^{r-2} \leq r!\mathrm{e}^r$, and the number of maps π by $r^\sigma \leq (\sigma-2)^\sigma$. Using the combinatoric identity (see [9] for a proof),

$$\sum_{\substack{\{I_j\}_{j=1}^\sigma \in \Pi_\sigma(\{1,\ldots,m\}) \\ 1 \in I_1}} \Big(\prod_{j=1}^{\sigma} l_{I_j}^{|I_j|-1}\Big) = \binom{m-1}{\sigma-1} l^{m-\sigma}, \quad 2 \leq \sigma \leq m, \tag{5.33}$$

where, as previously, $\Pi_\sigma(\{1,\ldots,m\})$ denotes the set of all partitions $\{I_j\}_{j=1}^\sigma$ of $\{1,\ldots,m\}$ into σ non-empty subsets, we get

$$\begin{aligned}\widetilde{Q}_{m,\sigma}(\eta_1;\ldots;\eta_m \mid k) &\leq (\sigma-2)^\sigma \binom{m-1}{\sigma-1} l^{m-\sigma} C^{m-1} (1+C)^{l-l_1-m+1} (h\overline{\nu}_1 C)^k \\ &\times \sum_{r=1}^{\min\{k,\sigma-2\}} \mathrm{e}^r 2^{\alpha(r\sigma-1)} 2^{(1+\alpha)(k-r)} \Big|\{(k_i)_{i=1}^{\sigma+r-1} : \sum_{i=1}^{\sigma+r-1} k_i = k-r\}\Big| \max_{T_m \in \mathcal{T}_m} \overline{\nu}_{T_m}.\end{aligned}$$

Summing over k and n, we have, setting $n' := n-k$ and $k' := k-r$,

$$\begin{aligned}&\sum_{n=1}^\infty \frac{1}{n!} Q_{m,\sigma}(\eta_1;\ldots;\eta_m \mid n) \\ &\leq (\sigma-2)^\sigma \binom{m-1}{\sigma-1} l^{m-\sigma} C^{m-1} (1+C)^{l-l_1-\sigma+1} h^l \sum_{k=1}^\infty (h\overline{\nu}_1 C)^k \sum_{n'=0}^\infty \frac{(h\nu_1)^{n'}}{n'!} N_{n'}^{(1)}(k+l) \\ &\quad \times \sum_{r=1}^{\min\{k,\sigma-2\}} 2^{(1+\alpha)(k-r)} 2^{\alpha(r\sigma-1)} \mathrm{e}^r \Big|\{(k_i)_{i=1}^{\sigma+r-1} : \sum_{i=1}^{\sigma+r-1} k_i = k-r\}\Big| \max_{T_m \in \mathcal{T}_m} \overline{\nu}_{T_m} \\ &\leq (\sigma-2)^\sigma \binom{m-1}{\sigma-1} l^{m-\sigma} C^{m-1} (1+C)^{l-l_1-\sigma+1} \sum_{r=1}^{\sigma-2} (h\overline{\nu}_1 C)^r \left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^{l+r} \\ &\quad \times 2^{\alpha(r\sigma-1)} \mathrm{e}^r \sum_{k'=0}^\infty \left(\frac{h\overline{\nu}_1 2^{1+\alpha} C}{1-h\nu_1\mathrm{e}}\right)^{k'} \max_{T_m \in \mathcal{T}_m} \overline{\nu}_{T_m} \\ &\leq (\sigma-2)^\sigma \binom{m-1}{\sigma-1} 2^{\alpha(\sigma-1)^2} l^{m-\sigma} C^{m-1} (1+C)^{l-l_1-\sigma+1} \left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^l \left(\frac{1-h\nu_1\mathrm{e}}{1-h\nu_1\mathrm{e}-h\overline{\nu}_1 2^{1+\alpha} C}\right)^{\sigma-1} \\ &\quad \times \sum_{r=1}^\infty \left(\frac{h\overline{\nu}_1 C\mathrm{e}}{1-h\nu_1\mathrm{e}-h\overline{\nu}_1 2^{1+\alpha} C}\right)^r \max_{T_m \in \mathcal{T}_m} \overline{\nu}_{T_m}.\end{aligned}$$

Here, we assumed that $h[\nu_1\mathrm{e} + \overline{\nu}_1 C(\mathrm{e} + 2^{1+\alpha})] < 1$. Theorem 5.1 in the case $m \geq 4$ is proven by setting, for any $3 \leq \sigma \leq m$,

$$\begin{aligned}A_{m,\sigma} := (\sigma-2)^\sigma \binom{m-1}{\sigma-1} 2^{\alpha(\sigma-1)^2} l^{m-\sigma} C^m (1+C)^{l-l_1-\sigma+1} \left(\frac{h}{1-h\nu_1\mathrm{e}}\right)^l \\ \times \left(\frac{1-h\nu_1\mathrm{e}}{1-h\nu_1\mathrm{e}-h\overline{\nu}_1 2^{1+\alpha} C}\right)^\sigma \frac{h\overline{\nu}_1\mathrm{e}(1-h\nu_1\mathrm{e}-h\overline{\nu}_1 2^{1+\alpha} C)}{1-h\nu_1\mathrm{e}-h\overline{\nu}_1(\mathrm{e}+2^{1+\alpha})C}.\end{aligned} \tag{5.34}$$

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